1 Motivation

1.1 Darcy's Law

In 1856, French engineer Henri Darcy concluded a series of experiments by which he deduced a simple linear relationship between the instantaneous discharge rate through a porous material and the pressure gradient over a certain distance.

$$Q = k(\kappa)\Delta P$$

Where Q is instantaneous flow, $\Delta P = P_l - P_0$ is the pressure gradient and k is a function of permeability, κ. Such simple relationships are abundant: this formula is analogous to Ohm's Law of electrical current, Fourier's Law of heat conduction or Fick's law of diffusion across a permeable membrane. Preserved are qualities which we intuitively wish to embody: if there is no pressure gradient over a distance, no flow occurs; if a pressure gradient does exist, flow will occur from high to low pressures; the greater the pressure gradient, the greater the discharge rate; the discharge rate will be different through different materials, even if the pressure gradient is the same.

Though Darcy derived his formula experimentally, later, it was theoretically derived from the Navier-Stokes equation via homogenization.

$$Q = \frac{A\kappa}{\mu L}\Delta P$$

Where μ is the viscosity, L is length and A is the cross-sectional area to flow. Thus, Darcy's Law replaces the description of a heterogeneous material with a homogeneous one with an identical discharge rate for a particular pressure gradient.

1.2 Objective

Although more accurate methods do exist for computing permeability, limited data and resources may make the application of Darcy's law the most efficient option. Petroleum engineering and geosciences make extensive use of this relationship when approximation and educated hypothesis are satisfactory. As a general rule, for a hydrocarbon or groundwater resource to be exploitable without stimulation, the permeability must be greater than 100 mD, where 1 darcy is $10^{-12}m^2$. Attempting to calculate the true value of permeability for an area as vast as the site of an oil reservoir would be impractical, if not impossible. Rather, geoscientists will calculate the permeability of a handful of samples taken at different depths and estimate the order of magnitude, thereby generalizing for the site.

What we seek is an methodology for selecting a value for k, the function of permeability, from a small sampling of data which maximally represents the system as a whole.

2 Numerical Solutions and Methods

Ideally, we would have access not only to the data derived from experimentation, but also to knowledge of the underlying reality: a mapping of every particle in the system with which we could validate a mathematical model by re-running the experiment, controlling for κ and letting Q vary. This is an unreasonable request; however, we can simulate it by generating a geometry and solving for fluid flows across a pressure gradient. In work done by Graveleau[3], the finite element package, Comsol Metaphysics, was used in tandem with Matlab's statistical toolboxes to just this end. Large domains, as seen below, were defined by

$$f(x) = \begin{cases} -0.5 & -2.5 < x < -1.8 \\ -0.15x^4 - 0.13x^3 - 0.65x^2 + 0.3^x + 1.5 & -1.8 < x < 1.8 \\ 0.5 & 1.8 < x < 2.5 \end{cases}$$

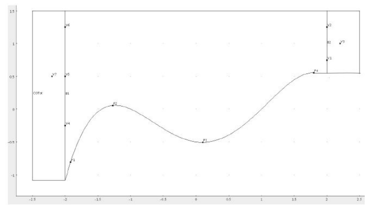
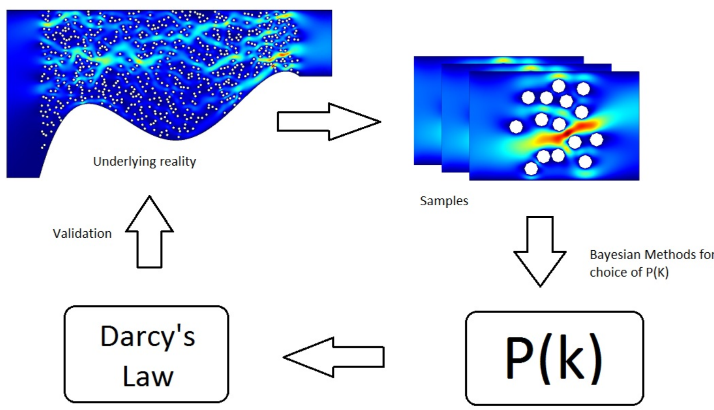


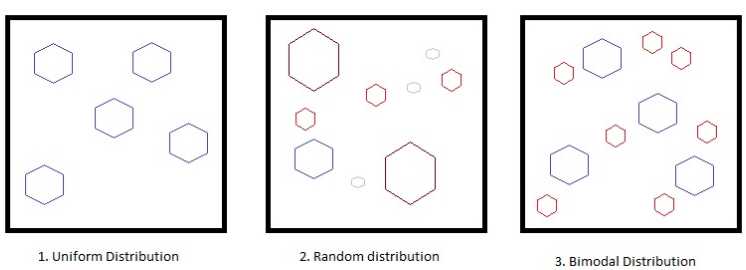
Figure 1: Defined Domain

Inside this surface, geometries were generated, allowing particles to overlap boundaries. Subjected to different pressure gradients, flow was computed at the end regions with *no-slip* boundary conditions (in which velocity is set to zero) at the top and bottom. From this domain, we sample small, rectangular units and perform similar computations. Analogous to sample matter collected in-field, we hope to determine what the large scale permeability will be by observing these units. Unlike experimentation, however, we will then be able to verify our claims by directly solving for flow. The process can be illustrated thusly



3 Methodology

For this report, three different geometries will be considered: uniformly distributed particles, randomly distributed particles and a mixture of two different particles, for each of which we will have five samples at three different pressure gradients (15 each, 45 total).



Although we have access to the underlying reality, for every sample we will behave as though the Comsol routine is a black box which outputs a set of data, Q, given three different pressure gradients, $\Delta P = 9; 999; 99,999$. These values were chosen for ease of computation, but also in an attempt to capture the linear relationship between pressure gradient and instantaneous flow. At large pressure gradients, $\delta P = 99,999$, we expect to see characteristics of turbulence.

We begin by the straightforward calculation of vector k,

$$k = Q^{-1}\Delta P$$

To get any further we must make hypothesis about the system. At the heart of Darcy's law is the idea of replacing of a heterogeneous medium for a homogenous one for which the flow is equivalent. Therefore we seek a single value for k. Yet we know this value will have an associated error when attempting to generalize about the whole system, therefore we seek a range of values that k might reasonably hold. This suggests a distribution. How should we approximate k? One approach would be to take the sample mean and variance, but we will see that this proves to be too simplistic after attempting to validate the model using the large-scale domain. It would be better to define k as coming from some distribution. And what to make of the vector Q? Does the experimenter have some knowledge of the margin of error in his apparatus, $sigma_e$? Is there a significant ratio between the error margins of k and the experimental error?

I will apply Bayesian statistical methods to the following scenarios:

- k is a parameter, σ_e is some known value
- k is a parameter, σ_e is a parameter
- k is normally distributed, σ_e is a parameter
- k is normally distributed, σ_k and σ_e are parameters
- k is derived from multiple Gaussians (After performing a chi-square test)

I will also attempt to make some hypothesis about the relationship between the σ_e and σ_k, and attempt to find the range of error the data allows, utilizing sampling of the posterior and bootstrap techniques.

4 References

1. Régis Cottereau, J. Tinsley Oden, Todd, A Oliver, Ernesto Preudencio and Serge Prudhomme. *Discussions on model errors and model validation.*2004.

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4. Freeze, Allen. *Henry Darcy and the Fountains of Dijon***Groundwater vol 32 no 1**. 1994 (23-30)

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