

Information Theory Blitz 2014

CS395T

All problems from Cover and Thomas, *Elements of Information Theory*.

Problem 1

A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

- (a) [2 points] Find the entropy $H(X)$ in bits.
- (b) [2 points] A random variable X is drawn according to this distribution. Find an “efficient” sequence of yes-no questions of the form, “Is X contained in the set S ?” Compare $H(X)$ to the expected number of questions required to determine X .

Problem 2

[2 point] Show that $H(Y|X) = 0$ if and only if Y is a deterministic function of X (i.e., for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$).

Problem 3

Let X_1 and X_2 be identically distributed but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

- (a) [1 point] Show that $\rho = \frac{I(X_1; X_2)}{H(X_1)}$.
- (b) [1 point] Show that $0 \leq \rho \leq 1$.
- (c) [1 point] When is $\rho = 0$?
- (d) [1 point] When is $\rho = 1$?

Problem 4

Let $p(x, y)$ be given by

$$p(X = 0, Y = 0) = p(X = 0, Y = 1) = p(X = 1, Y = 1) = \frac{1}{3}$$

and

$$p(X = 1, Y = 0) = 0.$$

Find:

- (a) [1 point] $H(X), H(Y)$.
- (b) [1 point] $H(X|Y), H(Y|X)$.
- (c) [1 point] $H(X, Y)$.

- (d) [1 point] $H(Y) - H(Y|X)$.
- (e) [1 point] $I(X; Y)$.
- (f) [1 point] Draw a Venn diagram for the quantities in parts (a) through (e).

Problem 5

The NBA finals is a seven-game series that terminates as soon as either team wins four games. Let X be the random variable that represents the outcome of a finals series between teams A and B. Some possible values of X are AAAA, BABABAB, and BBBAAAA. Let Y be the number of games played, which ranges from 4 to 7. Assuming that A and B are equally matched and that the games are independent, calculate

- (a) [2 points] $H(X), H(Y)$
- (b) [2 points] $H(Y|X)$
- (c) [2 points] $H(X|Y)$

Problem 6

[2 points] Let X have distribution $p(x)$, $x = 1, 2, \dots, m$. We are given a set $S \subseteq \{1, 2, \dots, m\}$. We ask whether $X \in S$ and receive the answer $Y = 1$ if it is and $Y = 0$ if it isn't. Suppose that $\mathbb{P}[X \in S] = \alpha$. Find the decrease in uncertainty $H(X) - H(X|Y)$.

Problem 7

[3 points] An urn contains r red and b black balls. Which has higher entropy, drawing $k \geq 2$ balls from the urn with replacement or without replacement?