Midterm Exam Solutions
March 3, 2014

1) a) $P(X|Y) \equiv \dfrac{P(XY)}{P(Y)}$, so (ii) is (writing A for P(A) and (AB) for P(AB), etc.)

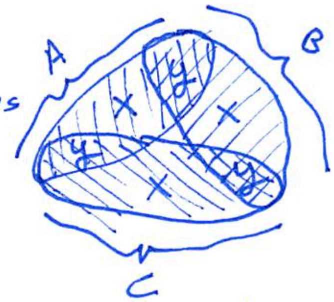
$$\frac{(AB)}{B} = \frac{(BC)}{C} = \frac{(CA)}{A} = \frac{(BA)}{A} = \frac{(CB)}{B} = \frac{(AC)}{C}$$

$\Rightarrow P(CA) = P(BA)$, other pairs $\Rightarrow$ all the $P(XY)$'s are equal.

then with numerators equal shows that $P(B) = P(C)$, other pairs show all the $P(X)$'s are equal, q.e.d.

(I could have chosen pairs like $\frac{(BC)}{C}$ and $\frac{(CB)}{B}$ to go directly to the requested result.)

b) The Venn diagram is thus



$P(B) = x + 2y$ e.g.

So $3x + 3y = 1$

$\Rightarrow x = \frac{1}{3} - y \geq 0$ *

So $P(B) = x + 2y = \left(\frac{1}{3} - y\right) + 2y = \frac{1}{3} + y$, also $= P(A), P(C)$

$\Rightarrow P(A)$ ranges from $\underline{1/3}$ when $y = 0$ to $\underline{\underline{\frac{2}{3}}}$ when $y = \frac{1}{3}$, which is y's max allowed value, from *

c) $\frac{1}{3} = P(B|C) = \dfrac{P(BC)}{P(C)} = \dfrac{y}{x+2y} = \dfrac{y}{\frac{1}{3}+y} \Rightarrow y = \frac{1}{6}$

So $P(A) = \frac{1}{3} + y = \underline{\underline{\frac{1}{2}}}$

2. a) $\left.\begin{aligned} P_X(x) &= \delta(x - x_0) \\ P_Y(y) &= \delta(y - y_0) \end{aligned}\right\}$ Dirac delta functions

b) $P_{X+Y}(z) = \int P_X(z - s) P_Y(s)\, ds$

$= \int \delta(z - s - x_0)\, \delta(s - y_0)\, ds$

I can pick either delta function to eliminate the integral say, this one

$$= \delta(z - s - x_0)\Big|_{s=y_0}$$

$$= \delta(z - (x_0 + y_0))$$

So it is massed at $x_0 + y_0$, q.e.d.

c) $$\Phi_X(t) = \int_{-\infty}^{+\infty} e^{itx} \rho_X(x)\,dx = \int_{-\infty}^{+\infty} e^{itx}\,\delta(x - x_0)\,dx$$

$$= e^{itx_0}$$

Similarly $\Phi_Y(t) = e^{ity_0}$

Then, $\Phi_{X+Y} = e^{itx_0} e^{ity_0} = e^{it(x_0+y_0)}$

which we can immediately see as the characteristic function of $\delta(z - (x_0 + y_0))$, q.e.d.

3. a) The test function is zero if both parts of the null hypothesis are exactly satisfied and always increases with increased violation of either piece. So it is a good choice for "closeness to the model".

b) One-sided, because the extreme cases of deviation all correspond to big values of T, not small values.

c) $$A_{meas} - X_{1\,meas} - X_{2\,meas}$$

the three crossed-out terms cancel under the null hypothesis

$$= (\cancel{A} + \epsilon) - (\cancel{X_1} + \epsilon) - (\cancel{X_2} + \epsilon)$$

independent draws from $N(0,1)$

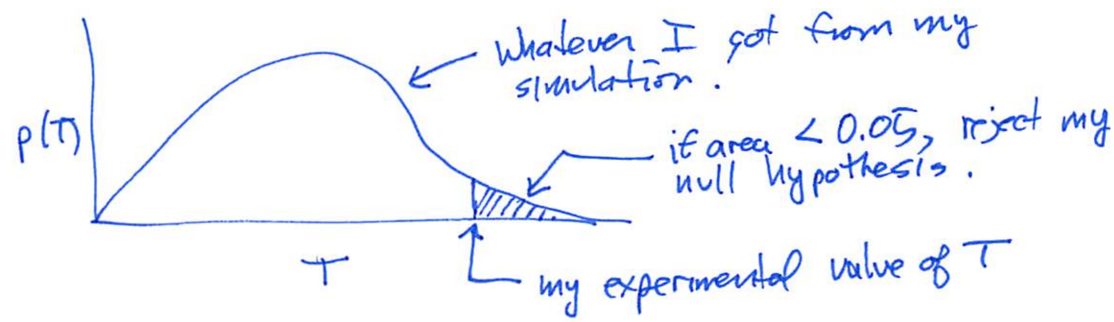
$$= \text{random variable drawn from } N(0, \sqrt{3}.)$$

Similarly for $B_{meas} - Y_{1\,meas} - Y_{2\,meas}$.

So I would simulate pairs of $N(0,\sqrt{3})$, square each one and sum the squares of the two to get a draw of T.

(Or, I might know that this is a χ^2 distribution!)

d)



4. $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\quad p(\underset{\sim}{X}) \propto e^{-\frac{1}{2}\underset{\sim}{X}\cdot\begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}\cdot\underset{\sim}{X}}$ ‡ (see below)

note each off-diagonal is half the multiplied-out cross term.

So $\Sigma^{-1} = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}$ but I need Σ.

Invert 2x2 matrix: ① $6X_1 + 4X_2 = b_1$
② $4X_1 + 3X_2 = b_2$

$3\times① - 4\times② \Rightarrow X_1 = \dfrac{3b_1 - 4b_2}{18-16} = \frac{3}{2}b_1 - 2b_2$

$4\times① - 6\times② \Rightarrow X_2 = \dfrac{4b_1 - 6b_2}{16-18} = -2b_1 + 3b_2$

$$\Rightarrow \Sigma = \begin{pmatrix} \frac{3}{2} & -2 \\ -2 & 3 \end{pmatrix}$$

a) mean is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since the quadratic form ‡ has $\underset{\sim}{X}$'s, not $(\underset{\sim}{X}-\underset{\sim}{\mu})$'s. Reading off from Σ,

$\underline{\underline{\mathrm{Var}(X_1) = \frac{3}{2}}}$, $\underline{\underline{\mathrm{Var}(X_2) = 3}}$, $\underline{\underline{\mathrm{Covar}(X_1, X_2) = -2}}$

b) constant is $\dfrac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}}$. Here $n=2$, $\det\Sigma = \frac{9}{2} - 4 = \frac{1}{2}$

so $= \dfrac{1}{2\pi\left(\frac{1}{2}\right)^{1/2}} = \underline{\underline{\dfrac{1}{\sqrt{2}\,\pi}}}$

c) Means + covariances are additive for independent variables, whether or not normal. So,

$\mu_Z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ $\quad \Sigma_Z = \begin{pmatrix} \frac{3}{2} & -2 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 7 & -4 \\ -4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} \frac{17}{2} & -6 \\ -6 & 6 \end{pmatrix}$$