**CSE383M and CS395T Midterm Exam**

March 3, 2014

1. The events A, B, and C are exhaustive but not mutually exclusive. That is, at least one event must occur in every trial. It is additionally observed that

(i) All three events never occur together, that is $P(ABC) = 0$, and

(ii) The six pairwise conditional probabilities are all equal. That is,

$$P(A|B) = P(B|C) = P(C|A) = P(B|A) = P(C|B) = P(A|C)$$

a) Prove that $P(A) = P(B) = P(C)$

b) What are the largest and smallest possible values for $P(A)$?

c) If $P(B|C)$ is 1/3, what is $P(A)$?

2. A random variable $X$ has a massed probability at a value $x_0$. (That is, it always has the value $x_0$. I admit that it is not a very interesting random variable, but that is just how it is!) Another random variable $Y$ has a massed probability at a value $y_0$.

a) Write the pdf's $P_X(x)$ and $P_Y(y)$ as functions. (If you are a stickler for terminology, then as generalized functions.)

Now consider the random variable

$$Z = X + Y$$

Obviously, $Z$ is massed at a value $x_0 + y_0$, but I want you to prove it as if it wasn't obvious.

b) Prove it using a formula that expresses the sum of two random variables as a convolution.

c) Prove it using characteristic functions.

3. You do an experiment that measures six real-value quantities, $A, x_1, x_2, B, y_1, y_2$. Your null hypothesis is that these should satisfy the relations,

$$A = x_1 + x_2 \qquad \text{and} \qquad B = y_1 + y_2$$

However, all six quantities are measured imperfectly, with normally distributed errors of mean zero and standard deviation 1.0. In other words,

$$A_{\text{measured}} = A_{\text{true}} + \epsilon, \quad \text{where} \quad \epsilon \sim N(0.0, 1.0)$$

and same for the other 5 measurements. A friend suggests that you use the test statistic

$$T = (A - x_1 - x_2)^2 + (B - y_1 - y_2)^2$$

a) Explain clearly why is this a good test statistic.

b) Should you use it one-sided or two-sided? Why?

c) How would you go about figuring out what is the distribution of the test statistic under the null hypothesis? (A bit later in the course we'll learn what it is analytically, but I'm not assuming that you already know this.)

d) How would you decide whether the null hypothesis is true?

4. Two random variables $x_1$ and $x_2$ have the joint p.d.f. given by

$$p(x_1, x_2) \propto \exp\left[-\tfrac{1}{2}(6x_1^2 + 8x_1x_2 + 3x_2^2)\right]$$

a) What are numerical values for the means, variances, and covariance: $\mu(x_1)$, $\mu(x_2)$, $\text{Var}(x_1)$, $\text{Var}(x_2)$, and $\text{Covar}(x_1, x_2)$?

b) What is the constant of proportionality (i.e., that turns "$\propto$" into "$=$") in the above p.d.f.?

Two other random variables $y_1$ and $y_2$, not necessarily normal, are independent of $x_1$ and $x_2$ and have the means and (co)variances

$$\mu\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 2. \\ 6. \end{bmatrix}, \qquad \text{Covar}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 7. & -4. \\ -4. & 3. \end{bmatrix}$$

c) What can you say about the mean and covariance of the new random variable

$$\mathbf{Z} \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}?$$