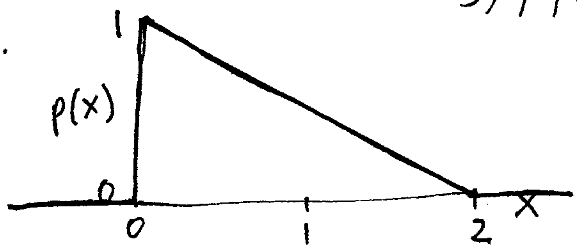
Surprise Quiz Solutions
3/4/2013

1.



2.
$$\mu = \int_0^2 x\, p(x)\, dx = \int_0^2 x\left(1-\tfrac{1}{2}x\right)dx = \frac{2}{3}$$

$$\langle x^2\rangle = \int_0^2 x^2 p(x)\,dx = \int_0^2 x^2\left(1-\tfrac{1}{2}x\right)dx = \frac{2}{3}$$

$$\sigma = \sqrt{\frac{2}{3} - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{2}}{3}$$

3.
$$CDF(x) = \begin{cases} 0 & x<0 \\ \int_0^x p(x')\,dx' = x - \frac{1}{4}x^2 & 0 \le x \le 2 \\ 1 & x>2 \end{cases}$$

4.
$$p = x - \tfrac{1}{4}x^2 \Rightarrow x = 2\left(1 - \sqrt{1-p}\right)$$

note choice of root to make x be between $0 + 2$!

So
$$\text{deviate} = 2\left(1 - \sqrt{\text{uniform deviate}}\right)$$

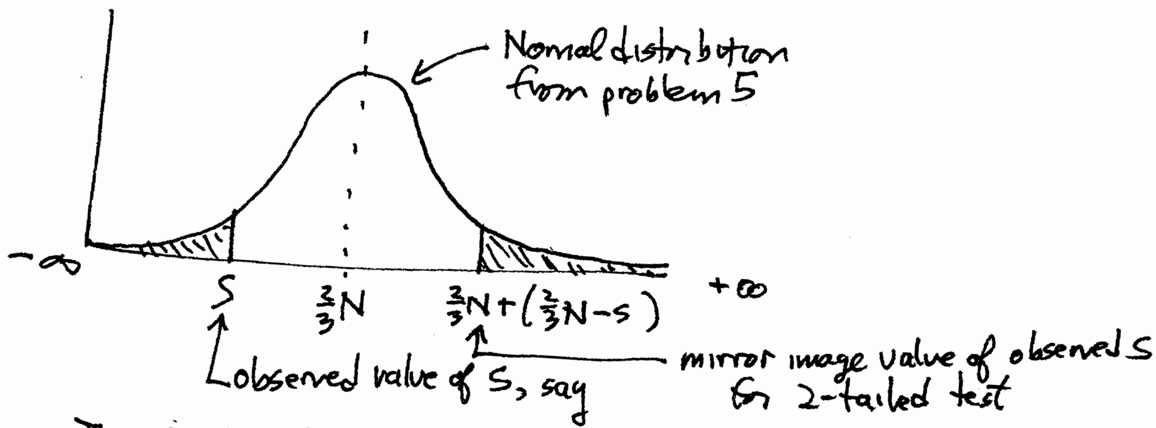
note that if p is uniform, so is $1-p$

5. It's normal (CLT) with N times the <u>mean</u> and N times the <u>variance</u> (not standard deviation!)

$$p(s) = \frac{1}{\sqrt{2\pi}\,\sqrt{N}\,\frac{\sqrt{2}}{3}} \exp\left[-\frac{1}{2}\left(\frac{s - N\cdot\frac{2}{3}}{\sqrt{N}\,\sqrt{2}/3}\right)^2\right]$$

$$= \frac{3}{\sqrt{4\pi N}} \exp\left[-\frac{9}{4}\,\frac{\left(s-\frac{2}{3}N\right)^2}{N}\right]$$

6.



The 2-tailed p-value is the sum of areas of the two shaded regions:

Compute these as $\int_{-\infty}^{s} \_\_\_ \, ds + \int_{\frac{2}{3}N+(\frac{2}{3}N-s)}^{\infty} \_\_\_ \, ds$ where ___ is problem 5 answer, and s is the observed sum.

Reject hypothesis if p-value less than 0.05 (or whatever)

7. Compute all 10 p-values as in problem 6.
Multiply each by 10 (Bonferroni correction)
Reject any of the 10 that are now > 0.05 (or whatever)
Remaining ones are not ruled out as correct hypotheses.

8. For each $j = 1 \ldots 10$, compute the probability of the data:

$$P_j \equiv P(\text{data} \mid H_j) = \prod_{i=1}^{28} p_X^{(j)}(x_i) \qquad \text{"test statistic"}$$

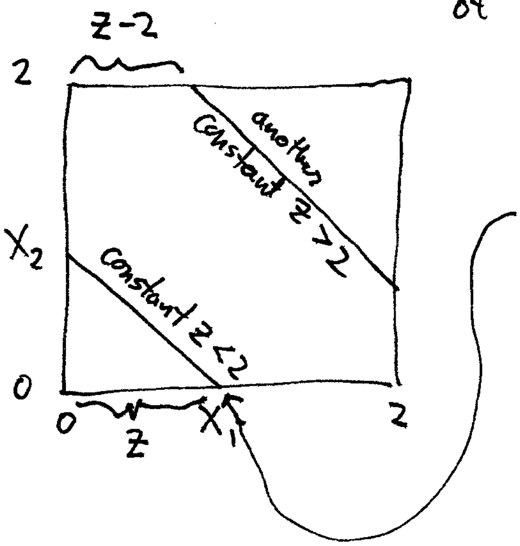
Assume uniform prior $P(H_j) = \frac{1}{10}$.

Bayes: largest P_j is most probable by a ratio (to any other hypothesis j') $P_j / P_{j'}$.

If these 9 ratios all large, one hypothesis is clearly favored.

9. $P_{X_1+X_2}(z) = \int p_X(x)\, p_X(z-x)\, dx$

The hard part is getting the limits of integration correct!



When $z<2$, we integrate x from 0 to z

But when $z>2$, we integrate x from $z-2$ to 2, as shown.

So

$$P_{X_1+X_2}(z) = \begin{cases} \int_0^z p(x')\,p(z-x')\,dx' = z - \frac{z^2}{2} + \frac{z^3}{24}, & 0 \le z \le 2 \\ \int_{z-2}^{2} p(x')\,p(z-x')\,dx' = \frac{8}{3} - 2s + \frac{s^2}{2} - \frac{s^3}{24}, & 2 \le z \le 4 \\ 0 & \text{otherwise} \end{cases}$$

oops: s should be z everywhere

Of couse I used Mathematica to do the integrals!

The next 2 pages show my Mathematica notebook for the whole exam.

(end of solutions)

In[1]:= `p[x_] := 1 - x / 2`

In[2]:= `mu = Integrate[x p[x], {x, 0, 2}]`

Out[2]= $\frac{2}{3}$

In[13]:= `mu2 = Integrate[x^2 p[x], {x, 0, 2}]`

Out[13]= $\frac{2}{3}$

In[3]:= `sig = Sqrt[Integrate[x^2 p[x], {x, 0, 2}] - mu^2]`

Out[3]= $\frac{\sqrt{2}}{3}$

In[4]:= `cdf = Integrate[p[x], {x, 0, s}]`

Out[4]= $s - \frac{s^2}{4}$

In[14]:= `Solve[pp == x - (1 / 4) x^2, x]` Inverse function of the CDF

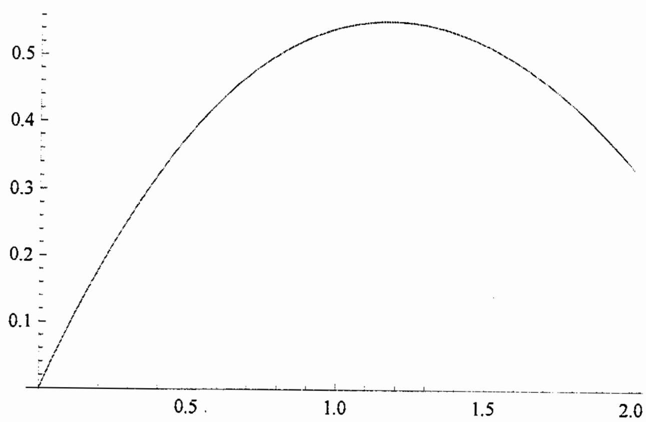
Out[14]= $\left\{\left\{x \to 2\left(1 - \sqrt{1 - pp}\right)\right\}, \left\{x \to 2\left(1 + \sqrt{1 - pp}\right)\right\}\right\}$

In[5]:= `pxy = Integrate[p[x] p[s - x], {x, 0, s}]` (s here is called z in the solution set)

Out[5]= $s - \frac{s^2}{2} + \frac{s^3}{24}$

In[6]:= `Plot[pxy, {s, 0, 2}]`

Out[6]=

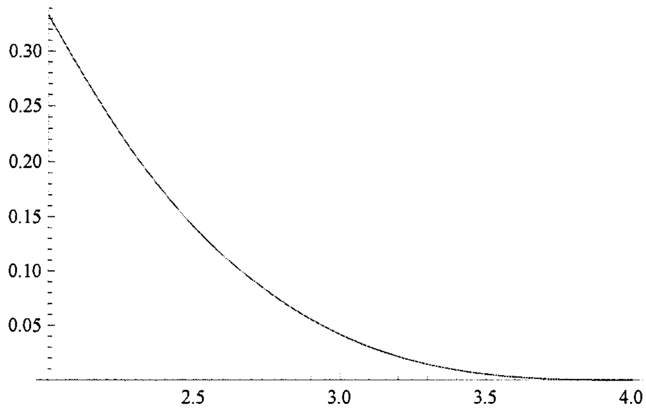


In[7]:= `pyx = Integrate[p[x] p[s - x], {x, s - 2, 2}]`

Out[7]= $\frac{8}{3} - 2 s + \frac{s^2}{2} - \frac{s^3}{24}$

In[8]:= `Plot[pyx, {s, 2, 4}]`

Out[8]=



In[9]:= `pxy /. s → 2`

Out[9]= $\frac{1}{3}$

check that the two answers match at $z = 2$

In[10]:= `pyx /. s → 2`

Out[10]= $\frac{1}{3}$

In[11]:= `Integrate[pxy, {s, 0, 2}] + Integrate[pyx, {s, 2, 4}]`

Out[11]= 1

+ check that they integrate to 1