

Midterm Exam Solutions
March 3, 2014

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80 points total
(points for each part indicated)

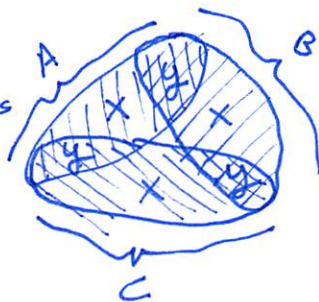
1) a) $P(X|Y) \equiv \frac{P(XY)}{P(Y)}$, so (ii) is (writing A for P(A) and (AB) for P(AB), etc.)

$$\frac{(AB)}{B} = \frac{(BC)}{C} = \frac{(CA)}{A} = \frac{(BA)}{A} = \frac{(CB)}{B} = \frac{(AC)}{C}$$

then with numerators equal shows that $P(B) = P(C)$,
other pairs show all the $P(X)$'s are equal, g.e.d.

(I could have chosen pairs like $\frac{(BC)}{C}$ and $\frac{(CB)}{B}$ to go directly to the requested result.)

b) The Venn diagram is thus



$$P(B) = x + 2y$$

e.g.

7 so $3x + 3y = 1$

$$\Rightarrow x = \frac{1}{3} - y \geq 0 \quad *$$

$$\text{So } P(B) = x + 2y = \left(\frac{1}{3} - y\right) + 2y = \frac{1}{3} + y, \text{ also } = P(A), P(C)$$

$\Rightarrow P(A)$ ranges from $\frac{1}{3}$ when $y=0$ to $\frac{2}{3}$ when $y = \frac{1}{3}$,
which is y 's max allowed value, from *

c) $\frac{1}{3} = P(B|C) = \frac{P(BC)}{P(C)} = \frac{y}{x+2y} = \frac{y}{\frac{1}{3}+y} \Rightarrow y = \frac{1}{6}$

6 so $P(A) = \frac{1}{3} + y = \underline{\underline{\frac{1}{2}}}$

2. a) $P_X(x) = \delta(x-x_0)$
 $P_Y(y) = \delta(y-y_0)$ } Dirac delta functions

b) $P_{X+Y}(z) = \int P_X(z-s) P_Y(s) ds$
 $= \int \delta(z-s-x_0) \delta(s-y_0) ds$

I can pick either delta function to eliminate the integral
say, this one

$$= \delta(z - s - x_0) \Big|_{s=y_0} \quad p.2$$

$$= \delta(z - (x_0 + y_0))$$

So it is massed at $x_0 + y_0$, q.e.d.

c)
 7

$$\Phi_X(z) = \int_{-\infty}^{+\infty} e^{itz} p_X(x) dx = \int_{-\infty}^{+\infty} e^{itz} \delta(x - x_0) dx$$

$$= e^{itx_0}$$

Similarly $\Phi_Y(z) = e^{ity_0}$

Then, $\Phi_{X+Y} = e^{itx_0} e^{ity_0} = e^{it(x_0 + y_0)}$

which we can immediately see as the characteristic function of $\delta(z - (x_0 + y_0))$, q.e.d.

3. a)
 5 The test function is zero if both parts of the null hypothesis are exactly satisfied and always increases with increased violation of either piece. So it is a good choice for "closeness to the model".

b)
 5 One-sided, because the extreme cases of deviation all correspond to big values of T , not small values.

c)
 5
$$A_{\text{meas}} - X_{1, \text{meas}} - X_{2, \text{meas}}$$

$$= (\cancel{A} + \epsilon) - (\cancel{X_1} + \epsilon) - (\cancel{X_2} + \epsilon)$$

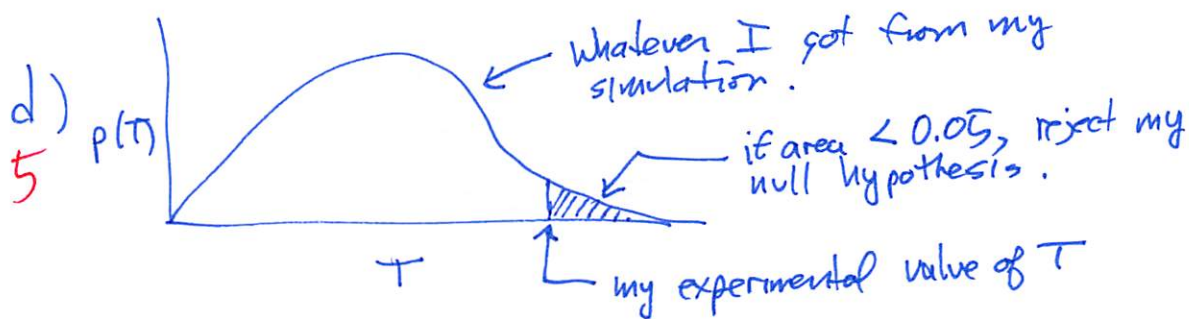
the three crossed-out terms cancel under the null hypothesis

independent draws from $N(0, 1)$

$$= \text{random variable drawn from } N(0, \sqrt{3}).$$

Similarly for $B_{\text{meas}} - Y_{1, \text{meas}} - Y_{2, \text{meas}}$.

So I would simulate pairs of $N(0, \sqrt{3})$, square each one and sum the squares of the two to get a draw of T . (Or, I might know that this is a χ^2 distribution!)



4. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $p(\underline{x}) \propto e^{-\frac{1}{2} \underline{x}^T \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix} \underline{x}}$ \neq (see below)

So $\Sigma^{-1} = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}$ but I need Σ .

↑ note each off-diagonal is half the multiplied-out cross term.

Invert 2×2 matrix:

- ① $6x_1 + 4x_2 = b_1$
- ② $4x_1 + 3x_2 = b_2$

$$3 \times \text{①} - 4 \times \text{②} \Rightarrow x_1 = \frac{3b_1 - 4b_2}{18 - 16} = \frac{3}{2}b_1 - 2b_2$$

$$4 \times \text{①} - 6 \times \text{②} \Rightarrow x_2 = \frac{4b_1 - 6b_2}{16 - 18} = -2b_1 + 3b_2$$

$$\Rightarrow \Sigma = \begin{pmatrix} \frac{3}{2} & -2 \\ -2 & 3 \end{pmatrix}$$

a) Mean is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since the quadratic form \neq has \underline{x} 's, not $(\underline{x} - \underline{\mu})$'s. Reading off from Σ ,

$$\text{Var}(x_1) = \frac{3}{2}, \text{Var}(x_2) = \underline{3}, \text{Covar}(x_1, x_2) = \underline{-2}$$

b) constant is $\frac{1}{(2\pi)^{n/2} (\det \Sigma)^{1/2}}$. Here $n=2$, $\det \Sigma = \frac{9}{2} - 4 = \frac{1}{2}$

$$\text{so } = \frac{1}{2\pi (\frac{1}{2})^{1/2}} = \frac{1}{\sqrt{2}\pi}$$

c) Means + covariances are additive for independent variables, whether or not normal. So,

$$\underline{\mu}_{\underline{z}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \Sigma_{\underline{z}} = \begin{pmatrix} \frac{3}{2} & -2 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 7 & -4 \\ -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{2} & -6 \\ -6 & 6 \end{pmatrix}$$