Opinionated Lessons in Statistics

by Bill Press

#9 Characteristic Functions

Characteristic Functions are a useful tool for understanding the sum of R.V.s

Statisticians often use notational convention that *X* is a random variable, *x* its value, $p_X(x)$ its distribution.

The characteristic function of a distribution is its Fourier transform.

$$\phi_X(t) \equiv \int_{-\infty}^{\infty} e^{itx} p_X(x) dx$$

$$\phi_X(0) = 1$$

$$\phi'_X(0) = \int ixp_X(x)dx = i \langle X \rangle$$

$$-\phi''_X(0) = \int x^2 p_X(x)dx = \mathrm{Var}(X) + \langle X \rangle^2$$

So, the coefficients of the Taylor series expansion of the characteristic function are the (uncentered) moments.

“The c.f. of the sum of independent r.v.’s is the product of their individual c.f.’s”

$$\text{let } S = X + Y$$

$$p_S(s) = \int p_X(u) p_Y(s-u)\, du$$

$$\phi_S(t) = \phi_X(t)\phi_Y(t)$$

Last line follows immediately from the Fourier convolution theorem. (In fact, it is the Fourier convolution theorem!)

Proof:

$$\phi_X(t) \equiv \int_{-\infty}^{\infty} e^{itx} p_X(x)dx$$

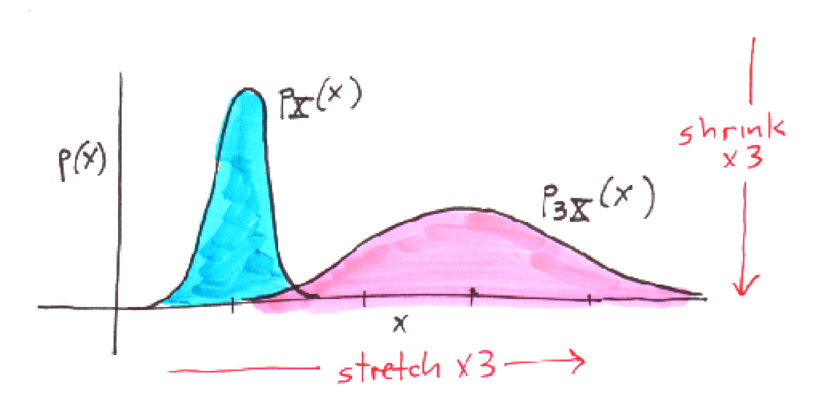
$$p_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(t)e^{-itx}dt$$

Fourier transform pair

$$\begin{aligned}
p_S(s) &= \int_{-\infty}^{\infty} p_X(u)p_Y(s-u)du \\
&= \int_{-\infty}^{\infty} p_X(u)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_Y(t)e^{-it(s-u)}dt\right]du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_Y(t)e^{-its}\left[\int_{-\infty}^{\infty}p_X(u)e^{itu}du\right]dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_Y(t)\phi_X(t)e^{-its}dt
\end{aligned}$$

So, $\phi_S(t) = \phi_Y(t)\phi_X(t)$

Scaling law for r.v.'s:



Scaling law for characteristic functions:

$$
\begin{aligned}
\phi_{aX}(t) &= \int e^{itx} p_{aX}(x) dx \\
&= \int e^{itx} \frac{1}{a} p_X\left(\frac{x}{a}\right) dx \\
&= \int e^{i(at)(x/a)} p_X\left(\frac{x}{a}\right) \frac{dx}{a} \\
&= \phi_X(at)
\end{aligned}
$$

What's the characteristic function of a Gaussian?

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In[14]:= $Assumptions = $Assumptions && (sig > 0)
```

Tell Mathematica that sig is positive. Otherwise it gives "cases" when taking the square root of sig^2

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In[15]:=
        p = (1 / (Sqrt[2 Pi] sig)) Exp[-(1 / 2) ((x - mu) / sig) ^2]
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Out[15]=

$$\frac{e^{-\frac{(-mu+x)^2}{2\,sig^2}}}{\sqrt{2\pi}\,sig}$$

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In[16]:= Integrate[p, {x, -Infinity, Infinity}]
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Out[16]=

1

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In[17]:= Integrate[p Exp[I t x], {x, -Infinity, Infinity}]
```

Out[17]=

$$e^{i\,mu\,t-\frac{sig^2\,t^2}{2}}$$

So the CF of a Gaussian is itself a Gaussian:

$$\phi_{\text{Normal}}(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

Cauchy distribution has ill-defined mean and infinite variance, but it has a perfectly good characteristic function:

Recall:

$$x \sim \text{Cauchy}(\mu, \sigma), \qquad \sigma > 0$$

$$p(x) = \frac{1}{\pi\sigma}\left(1 + \left[\frac{x-\mu}{\sigma}\right]^2\right)^{-1}$$

Matlab and Mathematica both (sadly) fail at computing the characteristic function of the Cauchy distribution, but you can use old-fashioned wetware methods (see proof posted on forum) to get:

$$\phi_{\text{Cauchy}}(t) = e^{i\mu t - \sigma|t|}$$

note non-analytic at t=0

Or, use social networking:

My Numerical Recipes co-author Saul says: "If t>0, close the contour in the upper 1/2-plane with a big semi-circle, which adds nothing. So the integral is just the residue at the pole $(x-\mu)/\sigma=i$, which gives $\exp(-\sigma t)$. Similarly, close the contour in the lower 1/2-plane for t<0, giving $\exp(\sigma t)$. So answer is $\exp(-|\sigma t|)$. The factor $\exp(i\mu t)$ comes from the change of x variable to $x-\mu$."