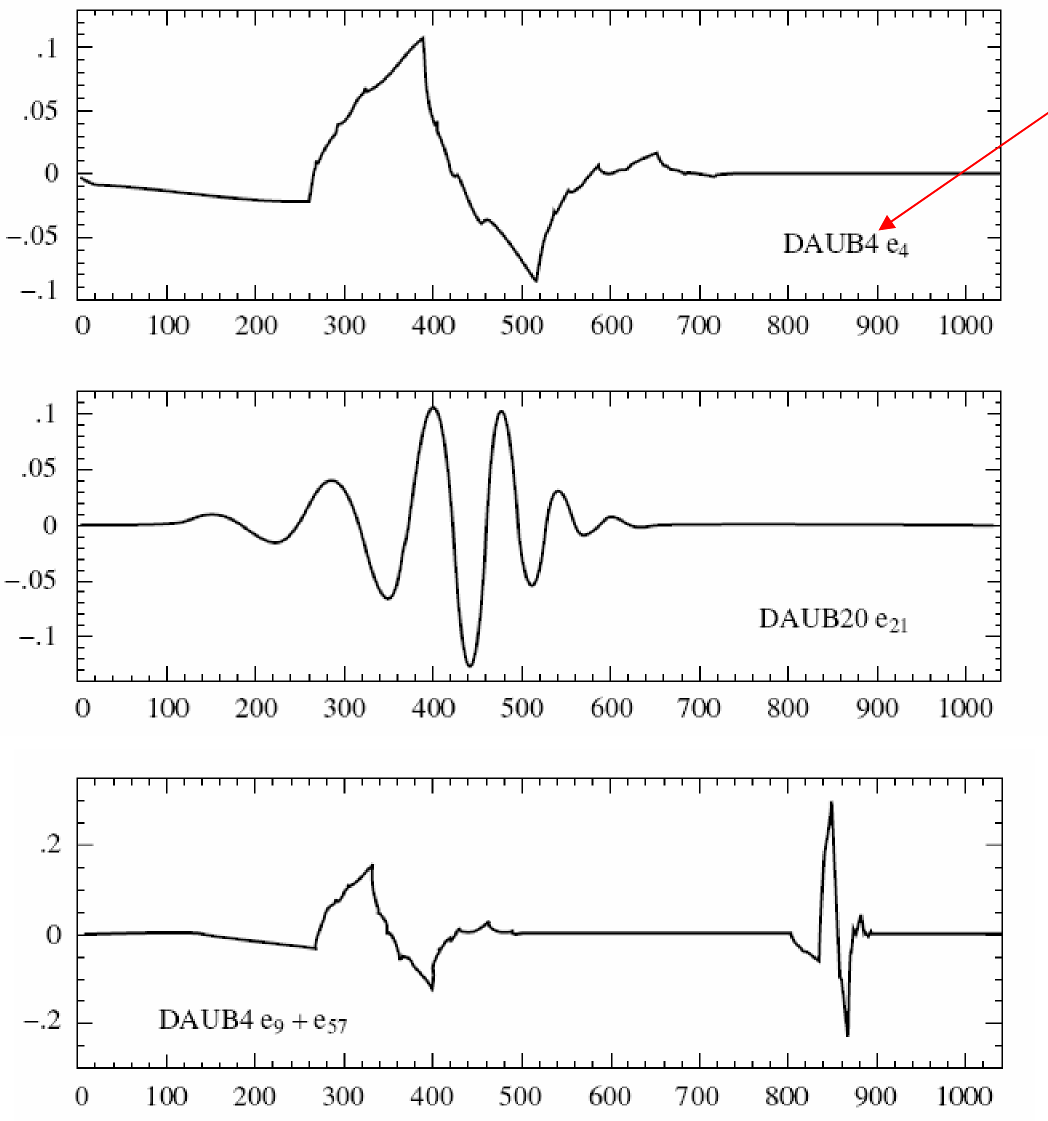
Opinionated Lessons in Statistics

by Bill Press

#44 Wavelets

Want to see some wavelets? Where do they come from?



The “DAUB” wavelets are named after Ingrid Daubechies, who discovered them.

(This is like getting the sine function named after you!)

So who is the sine function named after? it’s the literal translation into Latin, ca. 1500s, of the corres-ponding mathematical concept in Arabic, in which language the works of Hipparchus (~150 BC) and Ptolemy (~100 AD) were preserved. The tangent function wasn’t invented until the 9th Century, in Persia.

The first key idea in wavelets ("quadrature mirror filter") is to find an orthogonal transformation that separates "smooth" from "detail" information. We illustrate in the 1-D case.

$$
\begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & & & & & & & \\
c_3 & -c_2 & c_1 & -c_0 & & & & & & & \\
 & & c_0 & c_1 & c_2 & c_3 & & & & & \\
 & & c_3 & -c_2 & c_1 & -c_0 & & & & & \\
\vdots & \vdots & & & & & \ddots & & & & \\
 & & & & & & & c_0 & c_1 & c_2 & c_3 \\
 & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\
c_2 & c_3 & & & & & & & & c_0 & c_1 \\
c_1 & -c_0 & & & & & & & & c_3 & -c_2
\end{bmatrix}
$$

smooth average of 4 sequential components

not-smooth linear combination

transpose is

$$
\begin{bmatrix}
c_0 & c_3 & & & \cdots & & & & & c_2 & c_1 \\
c_1 & -c_2 & & & \cdots & & & & & c_3 & -c_0 \\
c_2 & c_1 & c_0 & c_3 & & & & & & & \\
c_3 & -c_0 & c_1 & -c_2 & & & & & & & \\
 & & & & \ddots & & & & & & \\
 & & & & & c_2 & c_1 & c_0 & c_3 & & \\
 & & & & & c_3 & -c_0 & c_1 & -c_2 & & \\
 & & & & & & & c_2 & c_1 & c_0 & c_3 \\
 & & & & & & & c_3 & -c_0 & c_1 & -c_2
\end{bmatrix}
$$

implying orthogonality conditions

$$
\begin{aligned}
c_0^2 + c_1^2 + c_2^2 + c_3^2 &= 1 \\
c_2 c_0 + c_3 c_1 &= 0
\end{aligned}
$$

these are two conditions on 4 unknowns, so we get to impose two more conditions

Choose the extra two conditions to make the not-smooth linear combination have zero response to smooth functions. That is, make its lowest moments vanish:

$$c_3 - c_2 + c_1 - c_0 = 0$$ no response to a constant function

$$0c_3 - 1c_2 + 2c_1 - 3c_0 = 0$$ no response to a linear function

The unique solution is now

$$c_0 = (1+\sqrt{3})/4\sqrt{2} \qquad c_1 = (3+\sqrt{3})/4\sqrt{2}$$
$$c_2 = (3-\sqrt{3})/4\sqrt{2} \qquad c_3 = (1-\sqrt{3})/4\sqrt{2}$$

"the DAUB4 wavelet coefficients"

If we had started with a wider-banded matrix we could have gotten higher order Daubechies wavelets (more zeroed moments), e.g., DAUB6:

$$c_0 = \left(1+\sqrt{10}+\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2} \qquad c_1 = \left(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2}$$
$$c_2 = \left(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2} \qquad c_3 = \left(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2}$$
$$c_4 = \left(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2} \qquad c_5 = \left(1+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right)/16\sqrt{2}$$

The second key idea in wavelets is to apply the orthogonal matrix multiple times, hierarchically. This is called the pyramidal algorithm.

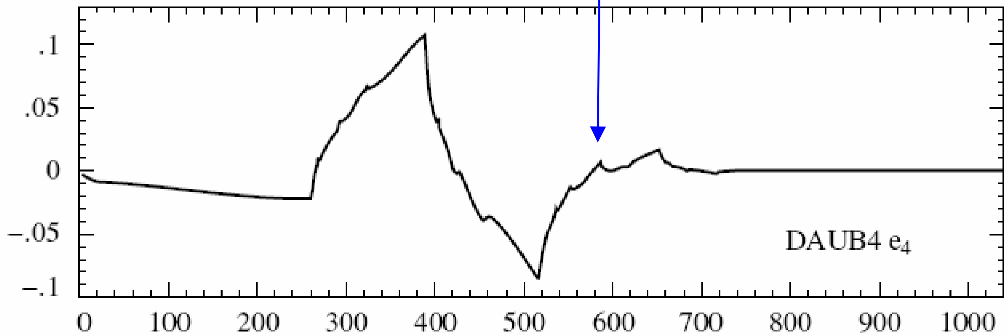
$$
\begin{bmatrix} y_0\\ y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8\\ y_9\\ y_{10}\\ y_{11}\\ y_{12}\\ y_{13}\\ y_{14}\\ y_{15} \end{bmatrix}
\xrightarrow{\text{QMF}}
\begin{bmatrix} s_0\\ d_0\\ s_1\\ d_1\\ s_2\\ d_2\\ s_3\\ d_3\\ s_4\\ d_4\\ s_5\\ d_5\\ s_6\\ d_6\\ s_7\\ d_7 \end{bmatrix}
\xrightarrow{\text{permute}}
\left[\begin{array}{c} s_0\\ s_1\\ s_2\\ s_3\\ s_4\\ s_5\\ s_6\\ s_7\\ \hline d_0\\ d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6\\ d_7 \end{array}\right]
\xrightarrow{\text{QMF}}
\left[\begin{array}{c} S_0\\ D_0\\ S_1\\ D_1\\ S_2\\ D_2\\ S_3\\ D_3\\ \hline d_0\\ d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6\\ d_7 \end{array}\right]
\xrightarrow{\text{permute}}
\left[\begin{array}{c} S_0\\ S_1\\ S_2\\ S_3\\ \hline D_0\\ D_1\\ D_2\\ D_3\\ \hline d_0\\ d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6\\ d_7 \end{array}\right]
\xrightarrow{\text{etc.}}
\left[\begin{array}{c} \mathcal{S}_0\\ \mathcal{S}_1\\ \hline \mathcal{D}_0\\ \mathcal{D}_1\\ \hline D_0\\ D_1\\ D_2\\ D_3\\ \hline d_0\\ d_1\\ d_2\\ d_3\\ d_4\\ d_5\\ d_6\\ d_7 \end{array}\right]
$$

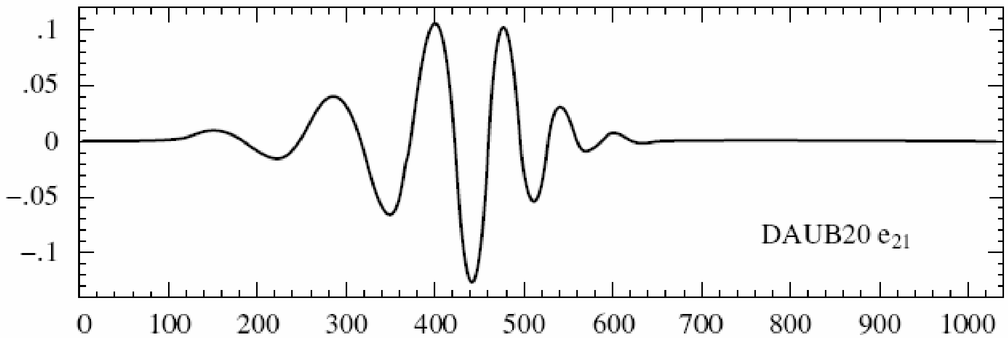
Since each step is an orthogonal rotation (either in the full space or in a subspace), the whole thing is still an orthogonal rotation in function space.

For multi-dimensional wavelet transforms, you separately transform each dimension, in any order. (Same procedure as multi-dimensional Fourier transform.)

We can see individual wavelets by taking the inverse transform of unit vectors in wavelet space:

the cusps are really there: DAUB4 has no right-derivative at values $p/2^n$, for integer p and n





Higher DAUBs gain about half a degree of continuity per 2 more coefficients. But not exactly half. The actual orders of regularity are irrational!

Continuity of the wavelet is not the same as continuity of the representation. DAUB4 represents piecewise linear functions exactly, e.g. But the cusps do show up in <u>truncated</u> representations as "wavelet plaid".

That's all for wavelets!