Opinionated Lessons in Statistics

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#15.5 Poisson Processes and Order Statistics
In a “constant rate Poisson process”, independent events occur with a constant probability per unit time.

In any small interval $\Delta t$, the probability of an event is $\lambda \Delta t$.

In any finite interval $T$, the mean (expected) number of events is $\lambda T$. 
What is the probability distribution of the waiting time to the 1\textsuperscript{st} event, or between events?

It’s the product of $t/\Delta t$ “didn’t occur” probabilities, times one “did occur” probability.

\[ p_{T_1}(t) \Delta t = [1 - \lambda \Delta t]^{t/\Delta t} \lambda \Delta t \]
\[ = e^{t \Delta t \ln(1 - \lambda \Delta t)} \lambda \Delta t \]
\[ \approx \lambda e^{-\lambda t} \Delta t \]

(random variable $T_1$ with value $t$)

\[ p_{T_1}(t) = \lambda e^{-\lambda t} = \lambda P_{\text{Poisson}}(0|\lambda t) \]

\[ t_1 \sim \text{Exponential}(\lambda) \]

Get it? It’s the probability that 0 events occurred in a Poisson distribution with $\lambda t$ mean events up to time $t$, times the probability (density) of one event occurring at time $t$. 
Once we understand the relation to Poisson, we immediately know the waiting time to the $k^{th}$ event:

\[ p_{T_k}(t) = \lambda P_{\text{Poisson}}(k-1|\lambda t) \]

\[ = \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \]

\[ = \frac{\lambda^k t^{k-1}}{\Gamma(k)} e^{-\lambda t} \]

$t_k \sim \text{Gamma}(k, \lambda)$

It’s the probability that $k-1$ events occurred in a Poisson distribution with $\lambda t$ mean events up to time $t$, times the probability (density) of one event occurring at time $t$.

We could also prove this with characteristic functions (sum of $k$ independent waiting times):

```
peXpon = lam Exp[-lam x]
```

```
pexponCF = Integrate[peXpon Exp[I t x],
{ x, 0, Infinity},
GenerateConditions -> False]
```

```
lam
lam - i t
```

```
pgamm = lam^k x^(k-1) Exp[-lam x] / Gamma[k]
```

```
pgammCF = Integrate[pgamm Exp[I t x],
{ x, 0, Infinity},
GenerateConditions -> False]
```

```
lam^k (lam - i t)^-k
```

$k^{th}$ power of above!
Same ideas go through for a “variable rate Poisson process”

Waiting time to first event or between events:

\[ p_{T_1}(t) = \lambda(t) \prod_j [1 - \lambda(t_j) \Delta t] \]

\[ = \lambda(t) e^{\sum_j \ln[1 - \lambda(t_j) \Delta t]} \lambda(t) \]

\[ \approx \lambda(t) e^{-\sum_j \lambda(t_j) \Delta t} \]

\[ = \lambda(t) e^{-\int_0^t \lambda(t) dt} \]

\[ \equiv \lambda(t) e^{-\Lambda(t)} \]

where \( \Lambda(t) \equiv \int_0^t \lambda(t) dt \)

so basically the area \( \Lambda(t) \) replaces the area \( \lambda t \)
Thus, waiting time for the kth event in a variable rate Poisson process is...

\[ p_{T_k}(t) = \lambda(t) \, P_{\text{Poisson}}[k - 1 | \Lambda(t)] \]

\[ = \lambda(t) \frac{[\Lambda(t)]^{k-1}}{(k-1)!} e^{-\Lambda(t)} \]

Notice the \( \lambda(t) \) – not \( \Lambda(t) \) – in front. So, in this form it is not Gamma distributed. We can recover the Gamma if we compute the probability density of \( \Lambda(t) \)

\[ p_{\Lambda_k}(\Lambda) = p_{T_k}(t) \frac{dt}{d\Lambda} = \frac{[\Lambda(t)]^{k-1}}{(k-1)!} e^{-\Lambda(t)} \]

So the “area (mean events) up to the kth event” is Gamma distributed,

\[ \Lambda_k \sim \text{Gamma}(k, 1) \]

How to simulate variable rate Poisson:
Draw from Gamma(1,1) [or Exponential(1) which is the same thing] and then advance through that much area under \( \lambda(t) \). That gives the next event.
What does this have to do with “order statistics”?

If \( N \) i.i.d. numbers are drawn from a univariate distribution, the \( k^{th} \) order statistic is the probability distribution of the \( k^{th} \) largest number.

Near the ends, with \( k \ll N \) or \( N - k \ll N \), this is just like variable-rate Poisson.

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How to simulate order statistics (approximation near extremes, large \( N \)):
Draw from Exponential(1) and then advance through that much area under the distribution of expected number of events (total area \( N \)). That gives the next event.
Order statistics: the exact result

The previous approximation is just the approximation of Binomial by Poisson (for large N and small k). Instead of what we had before,

\[ p_{T_k}(t) = \lambda(t) \, P_{\text{Poisson}}[k - 1 | \Lambda(t)] \]

a moment of thought gives the exact result,

\[ p_{T_k}(t) = \lambda(t) \, P_{\text{Binomial}}[k - 1 | N, \Lambda(t)/N] \]

Beta has the same relation to Binomial as Gamma (or Exponential) has to Poisson:

\[
\lim_{N \to \infty} N \text{Beta}(1,N) = \text{Exponential}(1)
\]
\[
\lim_{N \to \infty} N \text{Beta}(k,N) = \text{Gamma}(k, 1)
\]

How to simulate order statistics (exact):
Draw from Beta(1,N), giving a value between 0 and 1. Multiply by N. Advance through that much area under the distribution of expected number of events (total area N). That gives the next event. Decrement N by 1. Repeat.