Opinionated Lessons in Statistics

by Bill Press

#1 Let's talk about probability

Laws of Probability

"There is this thing called *probability*. It obeys the laws of an axiomatic system. When identified with the real world, it gives (partial) information about the future."

- What axiomatic system?
- How to identify to real world?
  - Bayesian or frequentist viewpoints are somewhat different "mappings" from axiomatic probability theory to the real world
  - yet both are useful

"And, it gives a consistent and complete calculus of inference."

- This is only a Bayesian viewpoint
  - It's sort of true and sort of not true, as we will see!
- R.T. Cox (1946) showed that reasonable assumptions about "degree of belief" uniquely imply the axioms of probability (and Bayes)
  - belief in a proposition's negation increases as belief in the proposition decreases
  - "composable" (belief in AB depends only on A and B|A)
  - belief in a proposition independent of the order in which supporting evidence is adduced (path-independence of belief)

Axioms:

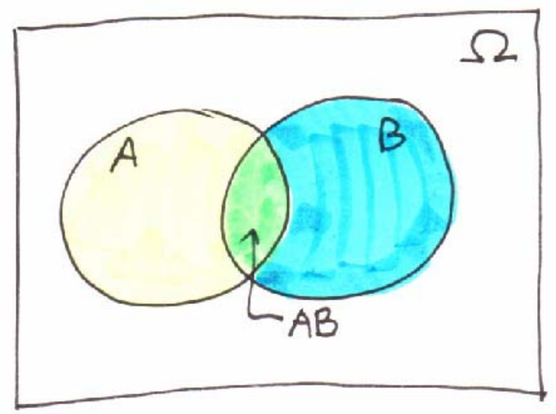
I. $P(A) \geq 0$ for an event A
II. $P(\Omega) = 1$ where Ω is the set of all possible outcomes
III. if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Example of a theorem:

Theorem: $P(\emptyset) = 0$
Proof: $A \cap \emptyset = \emptyset$, so
$P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$, q.e.d.

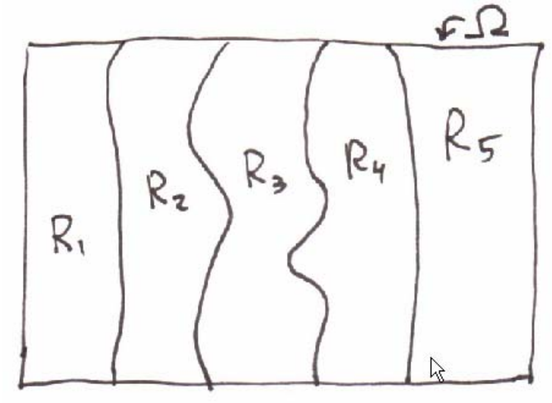
Basically this is a theory of measure on Venn diagrams, so we can (informally) cheat and prove theorems by inspection of the appropriate diagrams, as we now do.

Additivity or "Law of Or-ing"



$$P(A \cup B) = P(A) + P(B) - P(AB)$$

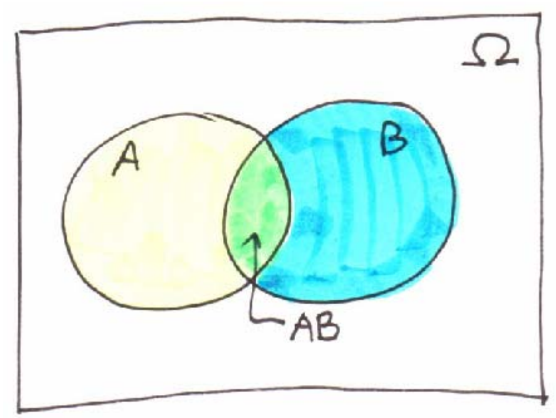
"Law of Exhaustion"



If R_i are exhaustive and mutually exclusive (EME)

$$\sum_i P(R_i) = 1$$

Multiplicative Rule or "Law of And-ing"



(same picture as before)

"given"

$$P(AB) = P(A)P(B|A) = P(B)P(A|B)$$

$$P(B|A) = \frac{P(AB)}{P(A)}$$

"conditional probability"

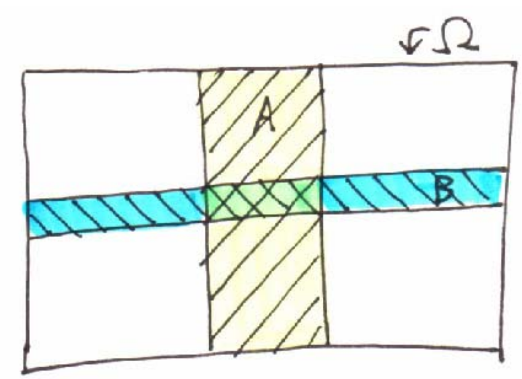
"renormalize the outcome space"

Similarly, for multiple And-ing:

$$P(ABC) = P(A)P(B|A)P(C|AB)$$

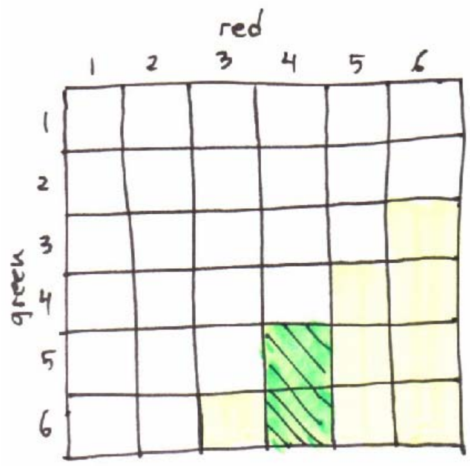
Independence:

Events A and B are independent if
$P(A|B) = P(A)$
so $P(AB) = P(B)P(A|B) = P(A)P(B)$



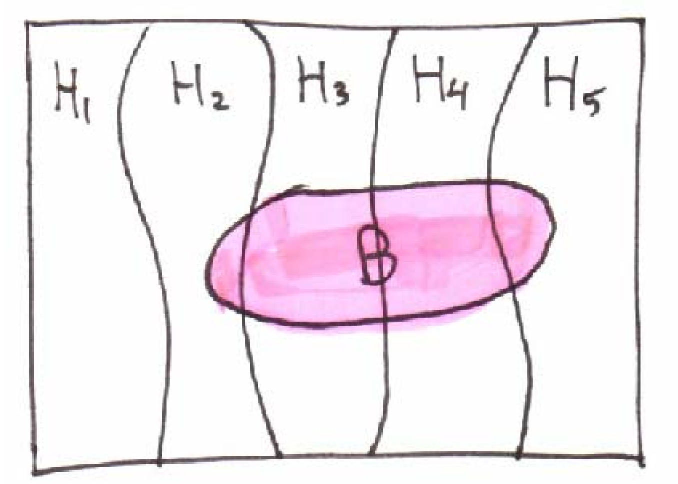
A symmetric die has
$P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$
Why? Because $\sum_i P(i) = 1$ and $P(i) = P(j)$.
Not because of "frequency of occurence in N trials".
That comes later!

The sum of faces of two dice (red and green) is > 8.
What is the probability that the red face is 4?



$$P(R4\,|\,{>}8) = \frac{P(R4 \cap {>}8)}{P({>}8)} = \frac{2/36}{10/36} = 0.2$$

Law of Total Probability or "Law of de-Anding"

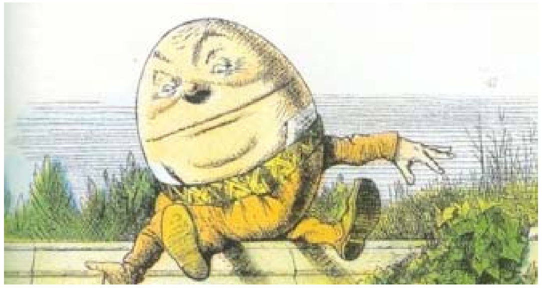


H's are exhaustive and mutually exclusive (EME)

$$P(B) = P(BH_1) + P(BH_2) + \ldots = \sum_i P(BH_i)$$

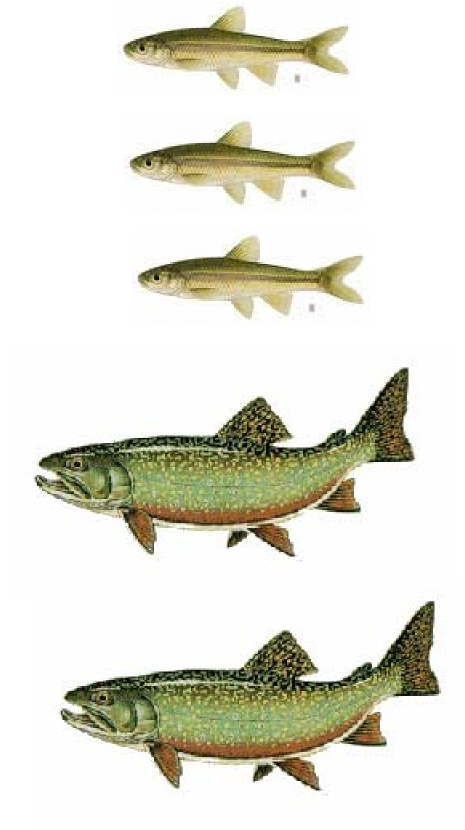
$$P(B) = \sum_i P(B|H_i)P(H_i)$$

"How to put Humpty-Dumpty back together again."



Example: A barrel has 3 minnows and 2 trout, with equal probability of being caught. Minnows must be thrown back. Trout we keep.

What is the probability that the 2nd fish caught is a trout?



$$
\begin{aligned}
H_1 &\equiv \text{1st caught is minnow, leaving } 3+2 \\
H_2 &\equiv \text{1st caught is trout, leaving } 3+1 \\
B &\equiv \text{2nd caught is a trout}
\end{aligned}
$$

$$
\begin{aligned}
P(B) &= P(B|H_1)P(H_1) + P(B|H_2)P(H_2) \\
&= \tfrac{2}{5}\cdot\tfrac{3}{5} + \tfrac{1}{4}\cdot\tfrac{2}{5} = 0.34
\end{aligned}
$$

Course preview question: About how many times would you have to do this experiment to distinguish the true value from a claim that P=1/3 ?