CS395T
Computational Statistics with Application to Bioinformatics

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Spring Term, 2011
The University of Texas at Austin

Lecture 25

Wiener Filtering (a.k.a. Optimal Filtering)

Norbert Wiener
1894 - 1964

This general idea can be applied whenever you have a basis in function space that concentrates "mostly signal" in some components relative to "mostly noise" in others.

You *could* just set components with too much noise to zero.

Wiener filtering is better: it gives the optimal way of tapering off the noisy components, so as to give the best (L^2 norm) reconstruction of the original signal.

Can be applied in spatial basis (delta functions, or pixels), Fourier basis (frequency components), wavelet basis, etc.

Different bases are not equivalent, because, in particular problems, signal and noise distribute differently in them. A lot of signal processing is finding the right basis for particular problems – in which signal is most concentrated.

(For simplicity, I'm going to write out the equations as if in a finite-dimensional space, but think infinite dimensional.)

You measure components of signal plus noise

$$C_i = S_i + N_i$$

Let's look for a signal estimator that simply scales the individual components of what is measured

$$\widehat{S}_i = C_i \Phi_i$$

$$\text{find the } \Phi_i\text{s that minimize } \left\langle |\widehat{\mathbf{S}} - \mathbf{S}|^2 \right\rangle$$

Here is where we use the fact that we are in some orthogonal basis, so the L^2 norm is just the sum of squares of components:

expectations come inside and land on the only two stochastic things, signal and noise

$$\left\langle (\widehat{\mathbf{S}} - \mathbf{S}) \cdot (\widehat{\mathbf{S}} - \mathbf{S}) \right\rangle = \left\langle \sum_i [(S_i + N_i)\Phi_i - S_i]^2 \right\rangle$$

$$= \sum_i \left\{ \left\langle S_i^2 \right\rangle (1 - \Phi_i)^2 + \left\langle N_i^2 \right\rangle \Phi_i^2 \right\} - 2 \sum_i \Phi_i (1 - \Phi_i) \left\langle N_i S_i \right\rangle$$

this is zero!

Differentiate w.r.t. Φ and set to zero, giving

$$\Phi_i = \frac{\left\langle S_i^2 \right\rangle}{\left\langle S_i^2 \right\rangle + \left\langle N_i^2 \right\rangle}$$

This is the Wiener filter. It requires estimates of the signal and noise power in each component.

Let's demonstrate in some different bases on this image:

```
IN = fopen('image-face.raw','r');
face = flipud(reshape(fread(IN),256,256)');
fclose(IN);
bwcolormap = [0:1/256:1; 0:1/256:1; 0:1/256:1]';
image(face)
colormap(bwcolormap);
axis('equal')
```

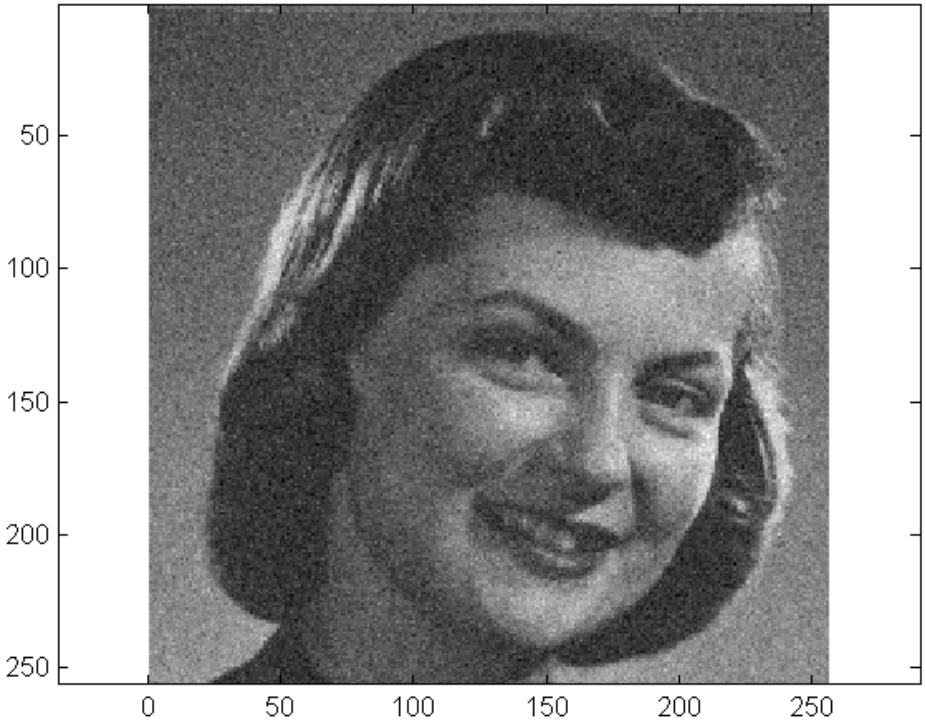
the file is unformatted bytes, so you read it with fopen and fread

(Favorite demo image in NR. Very retro, it's an IRE test photo from the 1950s, shows film grain and other defects.)

Add noise (here, Gaussian white noise):

```
noisyface = face + 20*randn(256,256);
noisyface = 255 * (noisyface - min(noisyface(:)))/(max(noisyface(:))-min(noisyface(:)));
image(noisyface)
colormap(bwcolormap);
axis('equal')
```

Have to rescale, because noise takes it out of 0-255.
Note reduced contrast resulting.



First example: **Fourier basis**

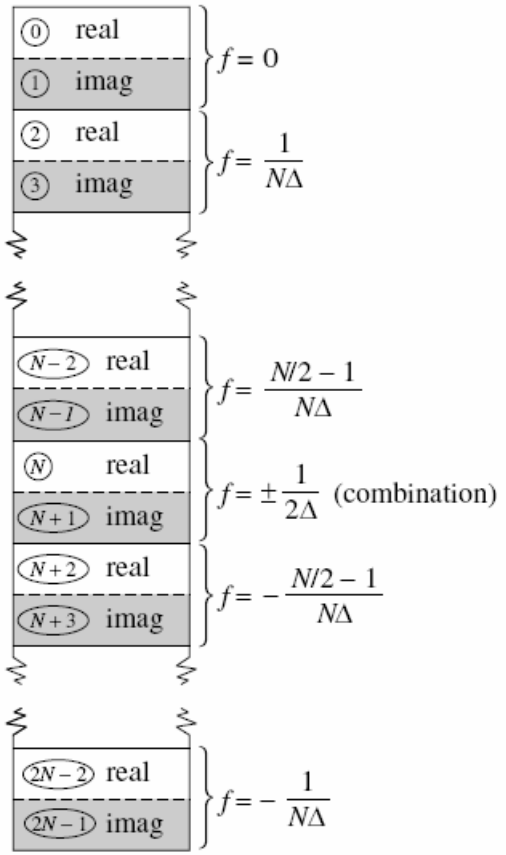
This will be a “low pass filter” using the fact that the signal is concentrated at low spatial frequencies, while the noise is white (flat).

Actually, Fourier is not a great basis for de-noising most images, since low-pass will reduce the resolution of the picture (blur it) along with de-noising.

```
fftface = fft2(noisyface);

[row col] = ndgrid(1:256);
ftable = [0:127,128:-1:1];
freqs = 0.5 * sqrt((ftable(row).^2+ftable(col).^2)/(2*128^2));
```

1 1 1… 1 2 3…
2 2 2… 1 2 3…
3 3 3… 1 2 3…

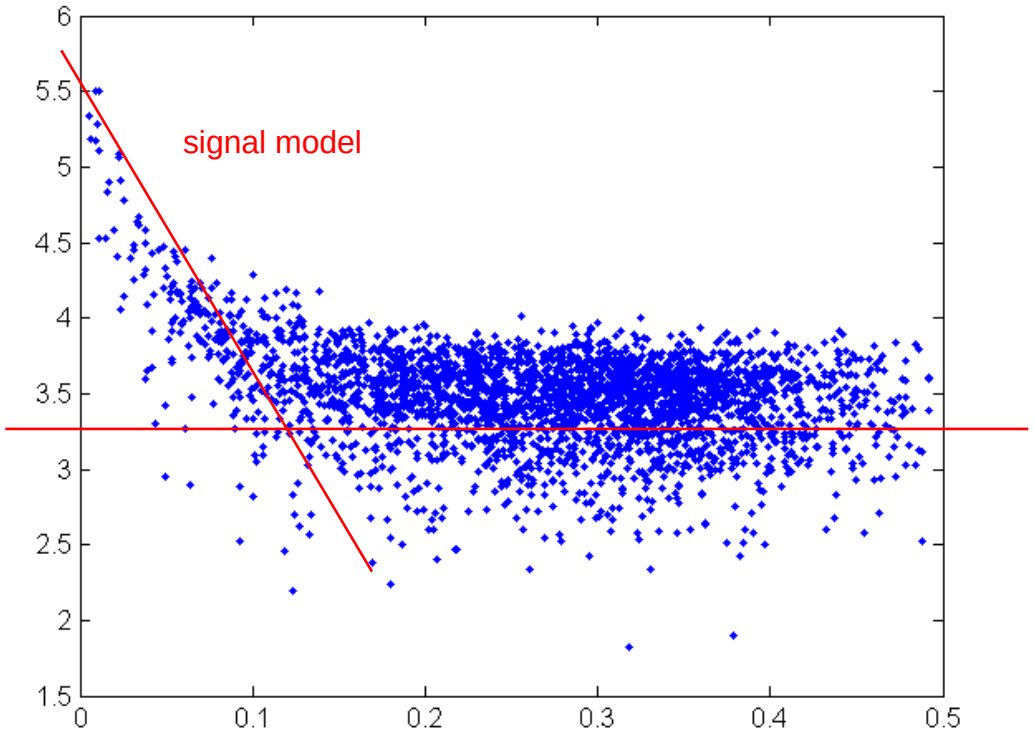
Nyquist frequency



Yes, we can see a separation between signal and noise:

```
samp = randsample(256*256,3000);
plot(freqs(samp),log10(abs(fftface(samp))),'.')
```

abs is here the complex modulus

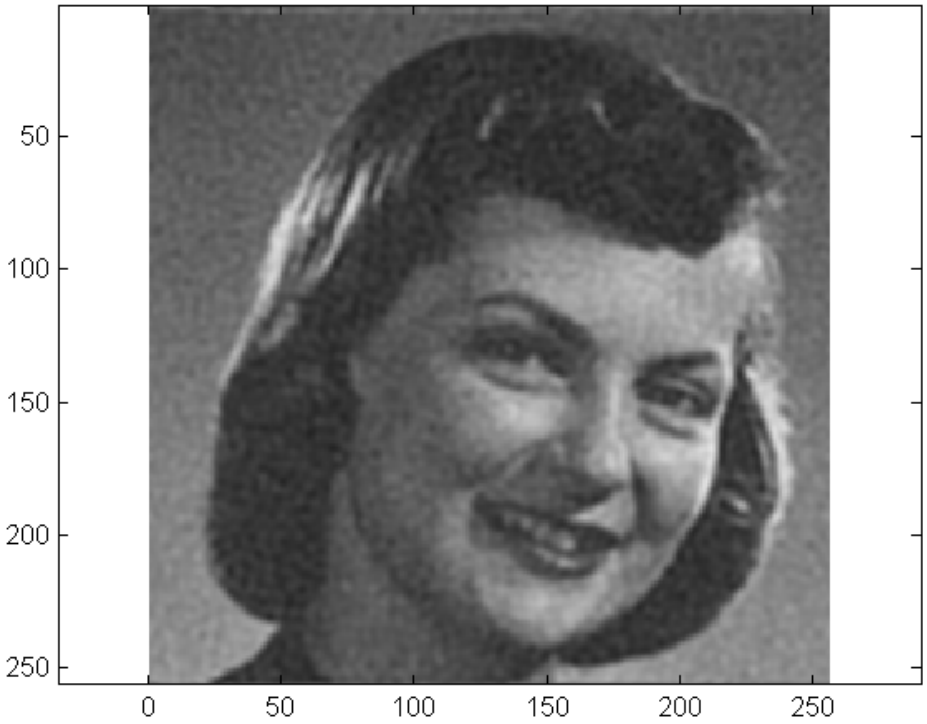


noise model

Why the line a bit low?
I played around with it to make the picture look better!

So my reconstruction is not exactly best least squares reconstruction, but will be less blurred.

de-noised



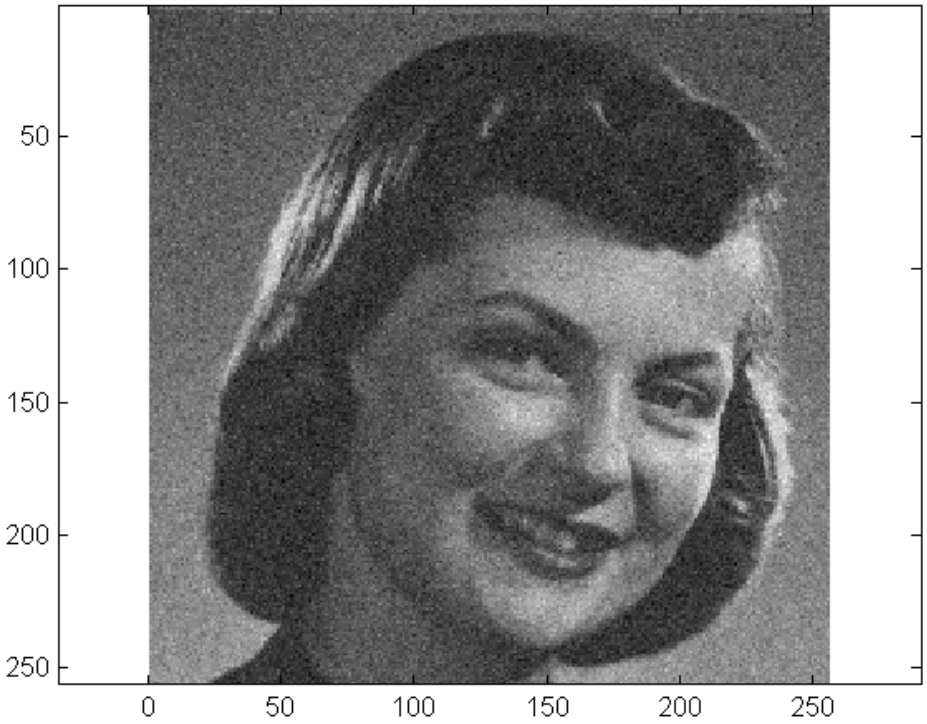

```
sig = 10.^(5.5 - 15 .* freqs);
noi = 10.^3.2;
fftfiltface = fftface .* (sig.^2 ./ (sig.^2 + noi.^2));
reface = ifft2(fftfiltface,'symmetric');
image(reface)
colormap(bwcolormap);
axis('equal')
```

I just read the constants off by eye from the previous chart

note square, to get power

this tells Matlab that you intend the inverse FFT to be real-valued

noisy



Actually, you might prefer the noisy image, because your brain has good algorithms for adaptively smoothing! But it is a less accurate representation of the original photo in L^2 norm!

Who was the 1950s IRE image lady? We may never know.

in November. The chairman reported that Subcommittee 3.2 on Methods of Measurement of Gain, Loss, Amplification, Attenuation and Amplitude-Frequency Response has been dissolved. R. C. Moody, Chairman of Subcommittee 3.3 on Methods of Measurement of Distortion, reported that his subcommittee is presently working on the definitions of intermodulation distortion and related measurement procedures. It is expected that a standards proposal on intermodulation distortion will be completed next spring. Mr. Maxwell announced the formation of a new Subcommittee 3.4 on Methods of Measurement of Noise. H. O. Saunders was appointed chairman of this subcommittee. Comments on the Proposed Standards on Audio Systems and Components: Methods of Measurement of Gain, Amplification, Loss, Attenuation, Ampli- under consideration by the Standards Committee. Mr. Redhead further reported that the Definitions on Storage Tubes will be submitted to the Standards Committee shortly. The Proposed Standards on Electron Tubes: Definitions of Terms Related to Gas Tubes was reviewed and modified by the committee, and referred to the subcommittee for approval of modifications.

K. R. McConnell presided at a meeting of the **Facsimile Committee** held at the Times Building, on September 16. L. R. Lankes submitted the first run photographic prints of the new IRE Facsimile Test Chart. The test chart was discussed by the committee at length. The comments on the Proposed Standards on Facsimile: Definitions of Terms were reviewed and corrections made. The proposed standards was approved for submission to the Standards Committee.

Feb., 1956

as chief automotive engineer. In 1945 he became associate laboratory manager of the oil company in New Jersey, the position he held until he came to this area in 1949 as a technical consultant and chief of the torpedo division at the Naval Ordnance Laboratory.

Dr. Schlesman held over twenty-three patents in petroleum and instrumentation.

He held membership in the American Chemical Society, the American Physical Society, Society of Automotive Engineers, American Association for the Advancement of Science, and American Petroleum Institute.

PROFESSIONAL GROUP NEWS

nents, only definitions and measurements.

Chairman W. R. Bennett presided at a meeting of the **Circuits** Committee held at IRE Headquarters on November 29. The entire meeting was devoted to the discussion of the Proposed Standard on Definitions that is being prepared by the committee.

The **Facsimile** Committee met at the Times Building on November 18 with Chairman K. R. McConnell presiding. The Committee reviewed the Proposed Standards on Facsimile: Definitions of Terms that is presently under consideration in the Standards Committee. The Proposed IRE Facsimile Test Chart was reviewed and corrections were made in the explanatory text that will accompany the Test Chart.

1957

IRE News and Radio Notes

Calendar of Coming Events and Author's Deadlines*

RETMA Symposium on Applied Reliability, Hotel Syracuse, Syracuse, N. Y., June 10–11

PGMIL Nat'l Meeting, Sheraton-Park Hotel, Washington, D. C., June 17–19

ACM Nat'l Meeting, Univ. of Houston, Houston, Tex., June 19–21

Brit. IRE Convention, Univ. of Cambridge, Eng., June 27–July 1

International Symposium on Physical Problems of Color Television, Paris, France, July 2–6 (DL*: May 1)

WESCON, Fairmont Hotel and Cow Palace, San Francisco, Calif., Aug. 20–23 (DL*: May 1)

URSI General Assembly, Boulder, Colo., Aug. 22–Sept. 5

Special Technical Conference on Magnetic Amplifiers, Penn Sheraton Hotel, Pittsburgh, Pa., Sept. 4–6 (DL*: June 11, D. Feldman, Bell Tel. Labs., Whippany, N. J.)

Industrial Electronics Symposium, Morrison Hotel, Chicago, Ill., Sept. 24–25

Nat'l Electronics Conference, Hotel Sherman, Chicago, Ill., Oct. 7–9 (DL*: June 1, V. H. Disney, Armour Res. Found., Chicago, Ill.)

IRE Canadian Convention, Exhibition Park, Toronto, Can., Oct. 16–18

East Coast Aero. & Nav. Conf., Lord Baltimore Hotel & 7th Reg. Armory, Balt., Md., Oct. 28–30

PGED Meeting, Shoreham Hotel, Wash., D. C., Oct. 31–Nov. 1 (DL*: Aug. 1, W. M. Webster, RCA, Somerville, N. J.)

PGNS Annual Meeting, Henry Hudson Hotel, New York City, Oct. 31–Nov. 1 (DL*: June 30, W. A. Higinbotham, Brookhaven Nat'l Labs., Upton, N. Y.)

Annual Symp. on Aero Commun., Hotel Utica, Utica, N. Y., Nov. 4–6

Radio Fall Meeting, King Edward Hotel, Toronto, Can., Nov. 11–13

PGI Conference, Atlanta-Biltmore Hotel, Atlanta, Ga., Nov. 11–13 (DL*: July 15, R. L. Whittle, Fed. Telecommun. Labs., 1389 Peachtree St., N.E., Atlanta 9, Ga.)

Mid-America Electronics Convention, Kan. City Mun. Audit., Kan. City, Mo., Nov. 13–14

PGVC Nat'l., Conf., Hotel Statler, Washington, D. C., Dec. 4–5. (DL*: July 1, G. E. Woodside, 1145 19th St., N.W., Wash., D. C.)

* DL = Deadline for submitting abstracts.

British IRE Holds Convention

The British Institution of Radio Engineers will hold its convention at the University of Cambridge from June 27 to July 1. The convention theme will be "Electronics in Automation" and thirty papers covering office machinery and information processing machine tool control, chemical and other processes, simulators, automation in the electronics industry, and automatic measurement and inspection will be presented during six sessions.

Guests attending the entire convention will be accommodated in King's College. Further details may be obtained from the British Institution of Radio Engineers, 9 Bedford Square, London, W. C. 1, England.

Engineering Education Note

It may be of interest to IRE members at large that the Board of Directors has approached the Engineers Council for Professional Development with the proposal that the IRE become a constituent member of that body.

Guided Missile Symposium Is Announced for Nov. 5–7

The annual Joint Military-Industry Guided Missile Reliability Symposium, sponsored this year by the Office of the Assistant Secretary of Defense for Engineering and the Chief of the Bureau of Aeronautics, Navy Department, will be held at the Naval Air Missile Test Center, Point Mugu, California. Participants will meet in technical sessions on November 5 and 6. November 7 will be reserved for visits to the Center's technical and operational areas.

Organizations desiring to send delegates to the symposium should select delegates by August 15 since attendance will be limited. Persons desiring to attend must have a secret security clearance.

Individuals and organizations wishing to present papers should submit abstracts of approximately 250 words no later than June 15, 1957. It is desired that contributions bear a security classification not higher than confidential and that abstracts be unclassified, if possible. Papers are invited on the following subjects: component problems, mathematical treatment, plant techniques, and testing techniques. Authors of papers selected for review will be notified by July 1 and will be expected to submit completed papers by July 15 for final selection. Security classification and clearances for all papers to be presented should be obtained directly by the authors from their contracting officer or authorized representative, keeping in mind that foreign scientists may attend the symposium. It is planned to have papers preprinted and distributed to advance registrants to permit review of papers by those desiring to engage in discussions.

All communications concerning meetings should be addressed to: Commander, NAMTC, Reliability Symposium, Code CEN-1, U. S. Naval Air Missile Test Center, Point Mugu, Calif.

IRE Facsimile Test Chart Available From RETMA

A test chart has been in the course of preparation for the last few years under the direction of the Technical Committee on Facsimile of the IRE and arrangements have been made for its sale through RETMA. This chart contains a wide variety of patterns expressly designed for the testing of facsimile equipment and systems. Because of these characteristics it may also be useful in other arts. A reduced reproduction of the chart is shown below.

It is a high quality glossy photographic print made under carefully controlled conditions to assure uniformity between different prints. The overall size is $9\frac{1}{4} \times 12\frac{1}{4}$ inches with the test patterns limited to $8\frac{1}{2} \times 11$ inches to allow for trimming.

Patterns are provided for checking single and multiple line definition, readability of different type faces, skew, index of cooperation, transient characteristics, jitter and grouping.

It contains a 15 step density tablet from paper white to about 1.7 with the densities calibrated. The maximum tolerance is ± 0.05 density unit up to 0.32 density and ± 0.11 density unit up to 1.00 density.

Eight of the sections will give different types of modulation patterns for synchronous oscilloscope display when scanned with a 2.75π inch scanning line.

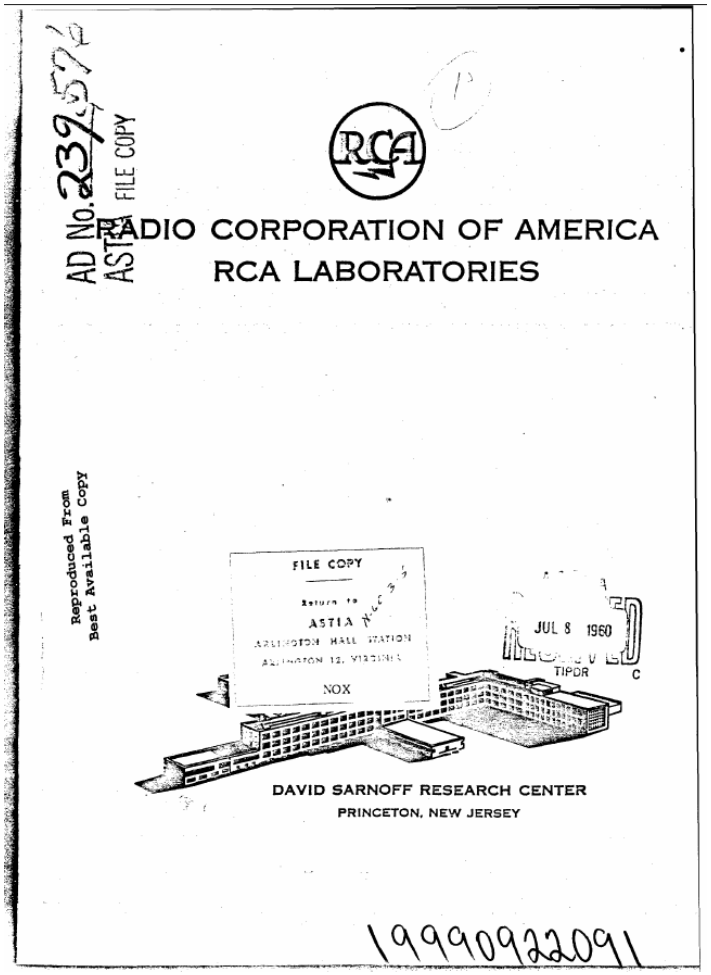
Instructions are furnished with the chart which identify and describe each pattern on the chart and indicate its use.

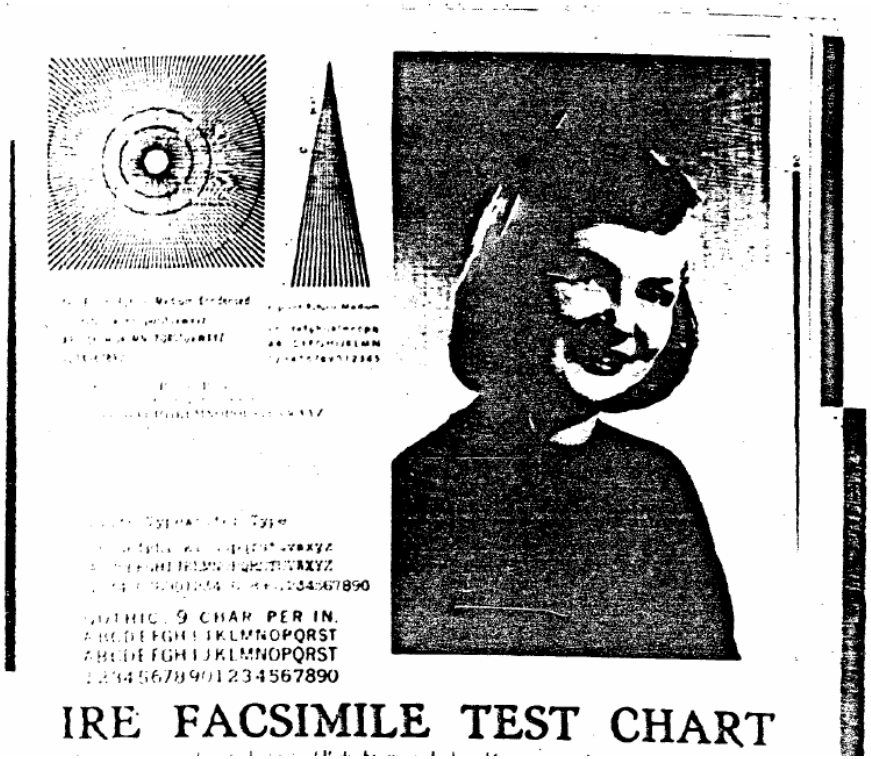
The IRE Facsimile Test Chart is available from the Radio-Electronics-Television Manufacturers' Association, Engineering Department, 11 West 42 Street, New York 36, N. Y., at the following prices:

Quantity	Copy of Instructions w/each Chart	Copy of Instructions w/each 10 Charts
1	$ 1.50	
10	10.50	$10.00
50	40.00	37.50
100	65.00	60.00

http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=4056616&userType=inst

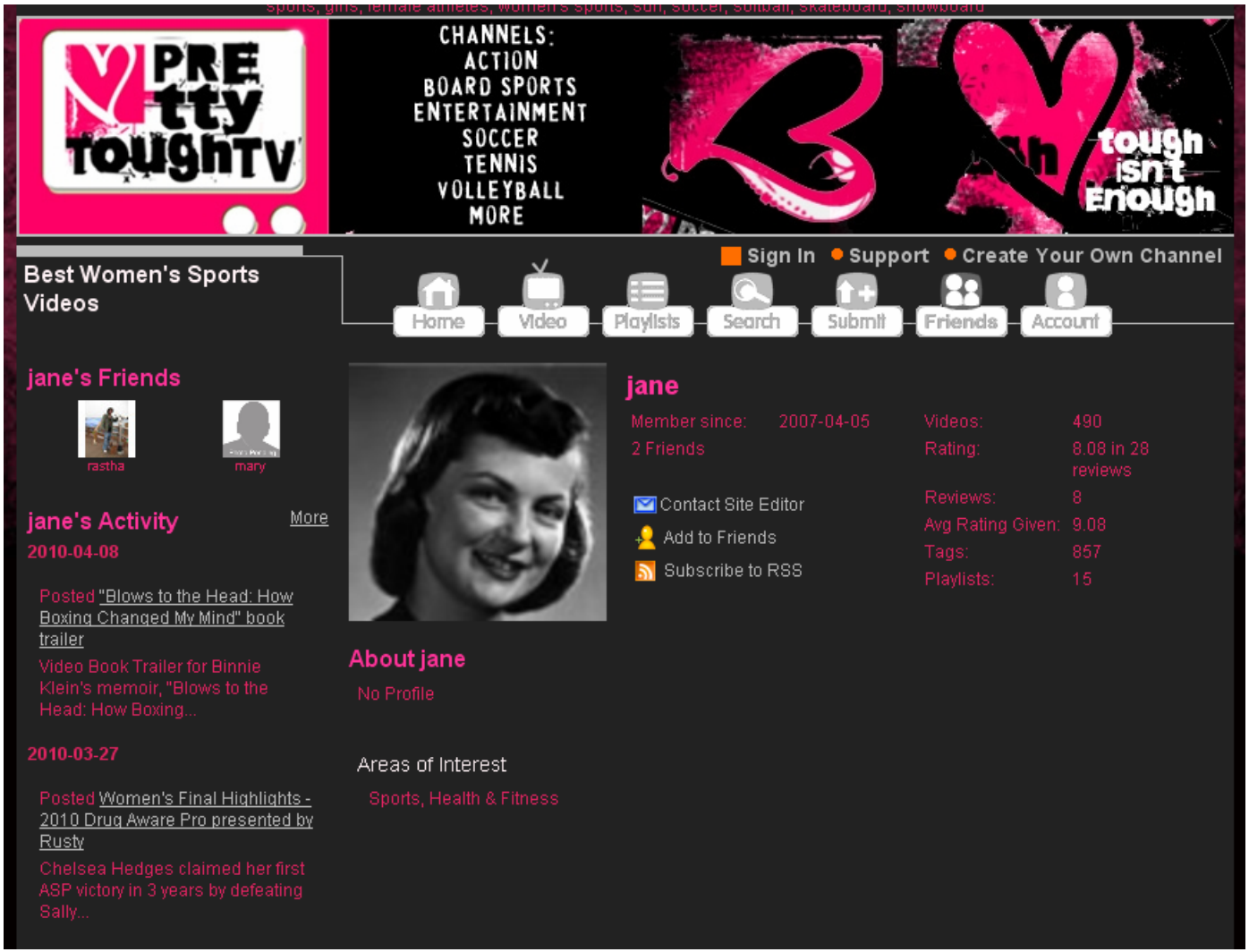
In 1960, she was briefly sighted in Princeton





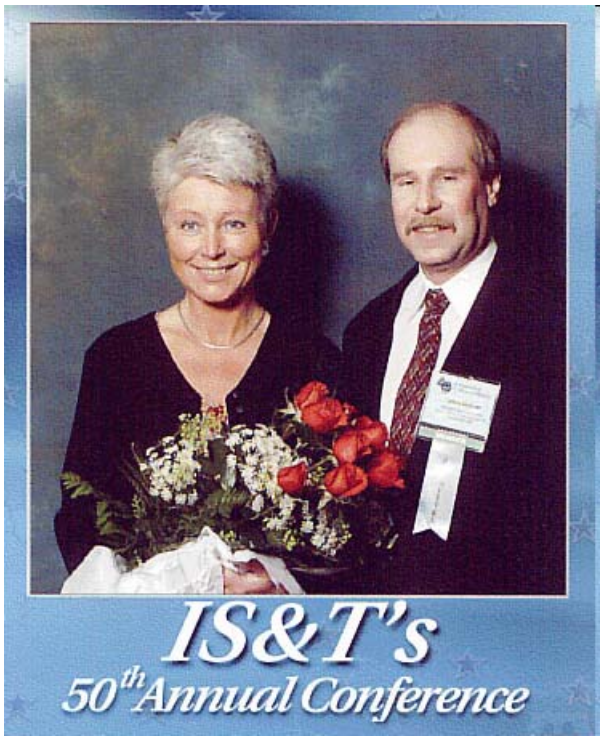
By 1987, when the IRE had already become the IEEE, she hadn't aged a bit!

Ageless, she was blogging in 2007, perhaps having been copied from NR!



We do know what happened to 1970s Lena, an early digital test image.

Lena, 1972 (detail)



Lena, 1997, accepting medal from Society for Imaging Science and Technology

Second example: **spatial (pixel) basis**

It doesn't make sense to use the pure pixel basis, because there is no particular separation of signal and noise separately in each pixel.

But a closely related method is to decompose the image into a smoothed background image, and then to take deviations from this as estimating signal power + noise power:

$$X_{ij} = \frac{1}{N_{\rm hood}} \sum_{\rm hood} x_{ij}$$

"hood" might be a 5x5 neighborhood centered on each point

$$\left\langle S_{ij}^2 + N_{ij}^2 \right\rangle = \frac{1}{N_{\rm hood}} \sum_{\rm hood} (x_{ij} - X_{ij})^2$$

this is the Wiener part

$$\widehat{x}_{ij} = X_{ij} + \frac{\left\langle S_{ij}^2 + N_{ij}^2 \right\rangle - \left\langle N_{ij}^2 \right\rangle}{\left\langle S_{ij}^2 + N_{ij}^2 \right\rangle} (x_{ij} - X_{ij})$$

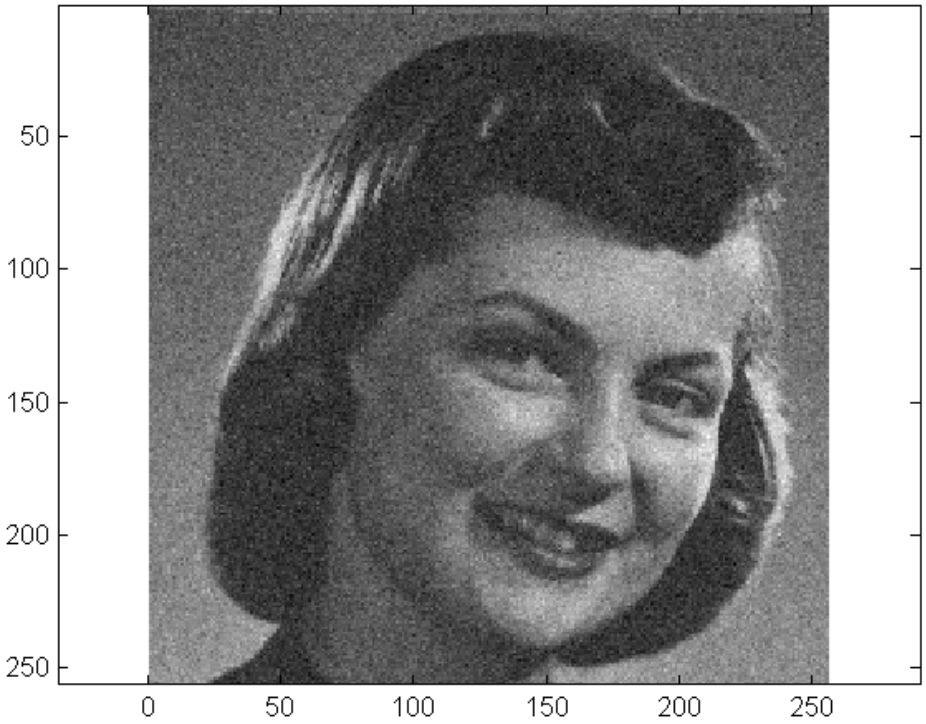
and let the user adjust $\left\langle N_{ij}^2 \right\rangle$ as a parameter.

Matlab has a function for this called wiener2


```
wiener = wiener2(noisyface,[5,5]);
image(wiener)
colormap(bwcolormap);
axis('equal')
```

you can put your noise estimate as another argument, or you can let Matlab estimate it as some kind of heuristic minimum of values seen for S^2+N^2 over the image

noisy



Most fun of all is the **wavelet basis**.
You don't even have to know what it is, except that it is an (orthogonal) rotation in function space, as is the Fourier transform.
(Its basis is localized both in space and in scale.)

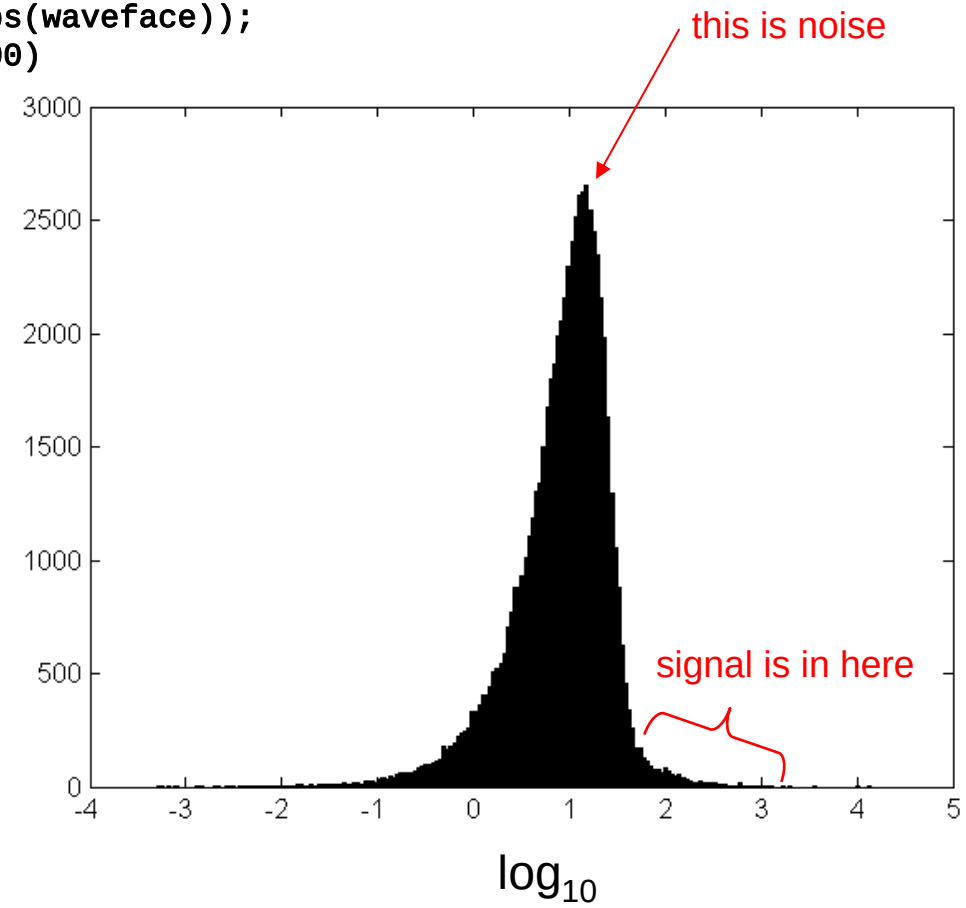
Matlab has a Wavelet Toolbox which I find completely incomprehensible! (I'm sure it's only me with this problem.) So, I'll do a mexfunction wrapper of the NR3 wavelet transform.

```
#include "nr3_matlab.h"
#include "wavelet.h"

/* Matlab usage:
        outmatrix = wavelet2(inmatrix,isign)
*/

void mexFunction(int nlhs, mxArray *plhs[], int nrhs, const mxArray *prhs[]) {
        MatDoub ain(prhs[0]);
        VecInt dims(2);
        Int mm=(dims[0]=ain.nrows()),nn=(dims[1]=ain.ncols());
        Int isign = Int(mxScalar<Doub>(prhs[1]));
        Int i,mn = mm*nn;
        Daub4 daub4;
        if (nrhs != 2 || nlhs != 1) throw("wavelet2.cpp: bad number of args");
        if ((nn & (nn-1)) != 0 || (mm & (mm-1)) != 0)
                throw("wavelet2.cpp: matrix sizes must be power of 2");
        VecDoub a(mn);
        for (i=0;i<mn;i++) a[i] = (&ain[0][0])[i];
        wtn(a,dims,isign,daub4);
        MatDoub aout(mm,nn,plhs[0]);
        for (i=0;i<mn;i++) (&aout[0][0])[i] = a[i];
        return;
}
```

this is the whole point,
the NR3 wavelet transform

Take the wavelet transform and look at the magnitude of the components on a log scale:

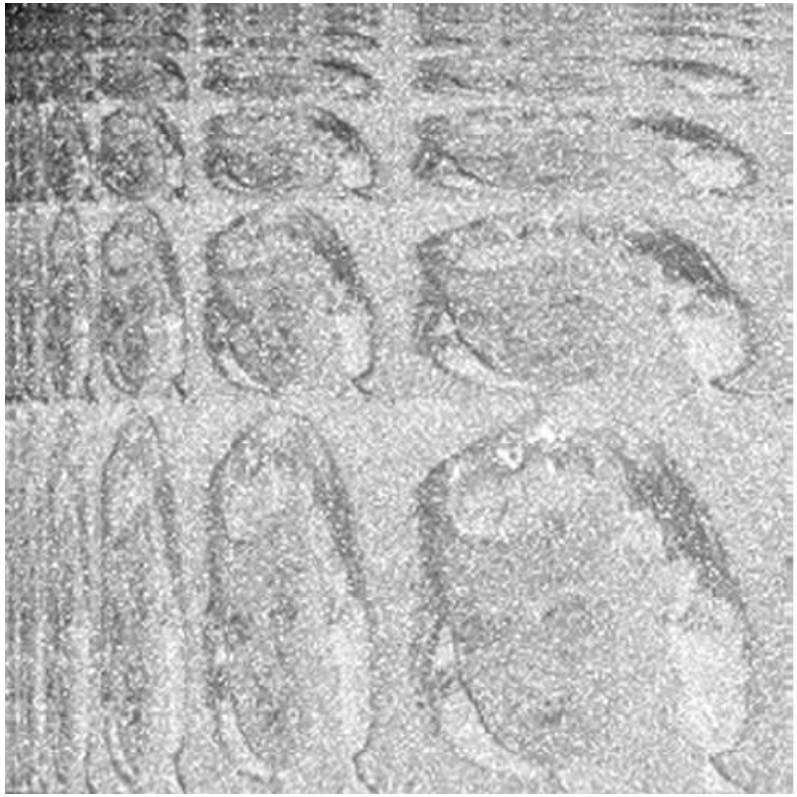
```
waveface = wavelet2(noisyface,1);
dist = log10(abs(waveface));
hist(dist(:),200)
```



Notice the difference in philosophy from Fourier: There we used frequency ("which component") to estimate S and N. Here we use the magnitude of the component directly , without regard to which component it is.

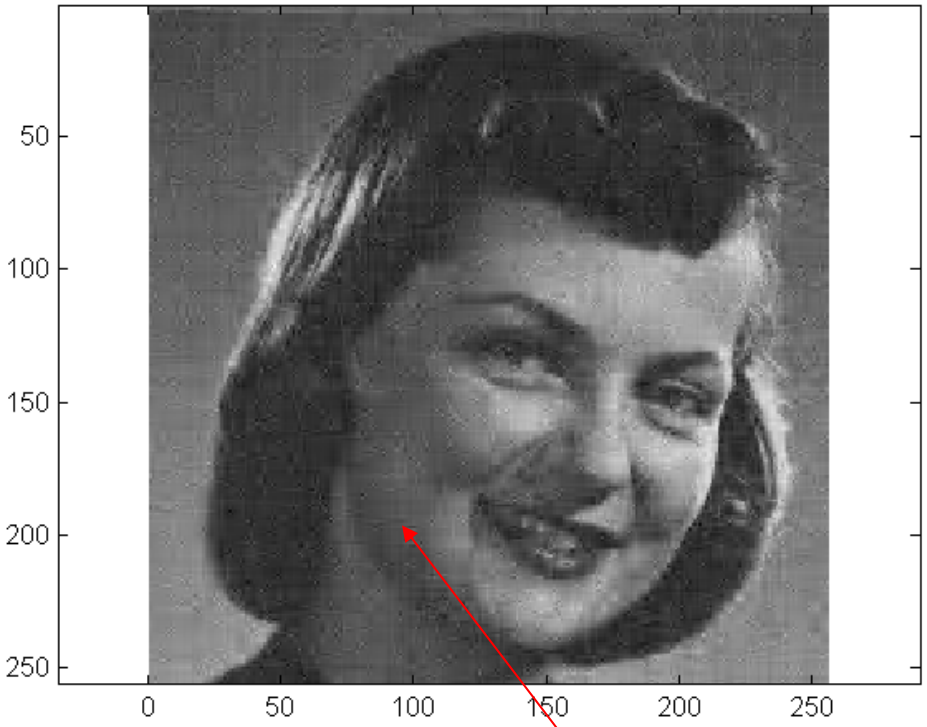
If you fiddle around with mapping the gray scale (zero point, contrast, etc.) of the matrix "`waveface`" you can see how the wavelet basis works

low resolution information is in this corner



high resolution information is in this corner

Truncate-to-zero components with magnitude less than 30.
This is not a true Wiener filter, because it doesn't roll off smoothly.




```
fwaveface = waveface;
fwaveface(abs(waveface)<30) = 0.;
werecface = wavelet2(fwaveface,-1);
image(werecface)
colormap(bwcolormap)
axis('equal')
```

Notice the "wavelet plaid" in the image. You sometimes see this on digital TV, because MPEG4 uses wavelets for still texture coding.

Compare to Wiener filter (smooth roll-off)

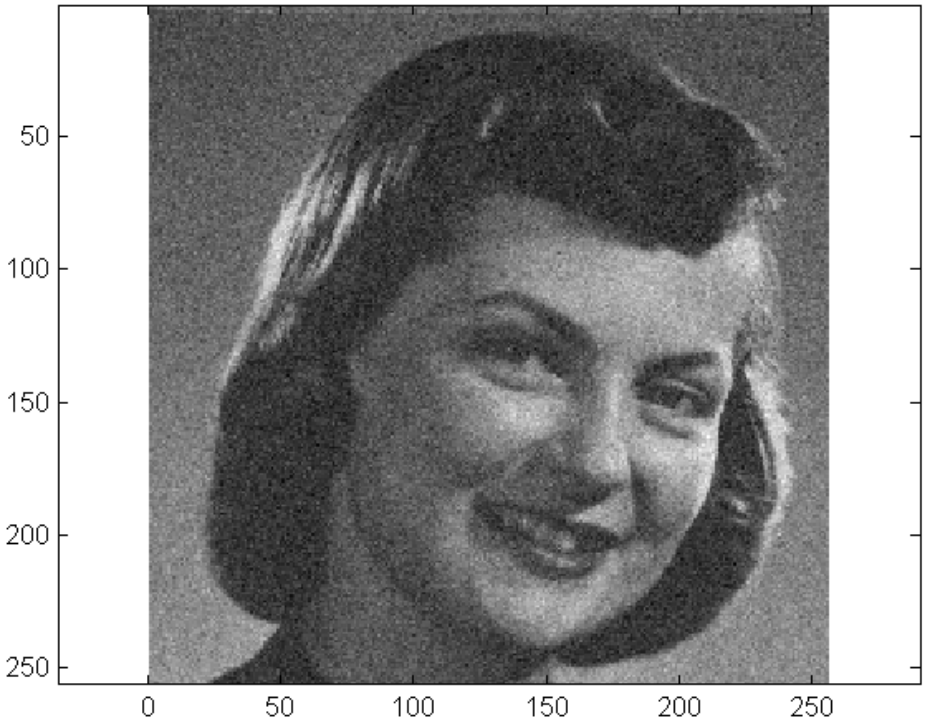

```
fwaveface = waveface .* (waveface .^ 2 ./ (waveface.^2 + 900));
werecface = wavelet2(fwaveface,-1);
image(werecface)
colormap(bwcolormap)
axis('equal')
```

i.e., noise amplitude 30 in the previous histogram

Even better if we restore the contrast


```
werecface = 255*(werecface - min(werecface(:)))/(max(werecface(:))-min(werecface(:)));
image(werecface)
colormap(bwcolormap)
axis('equal')
```

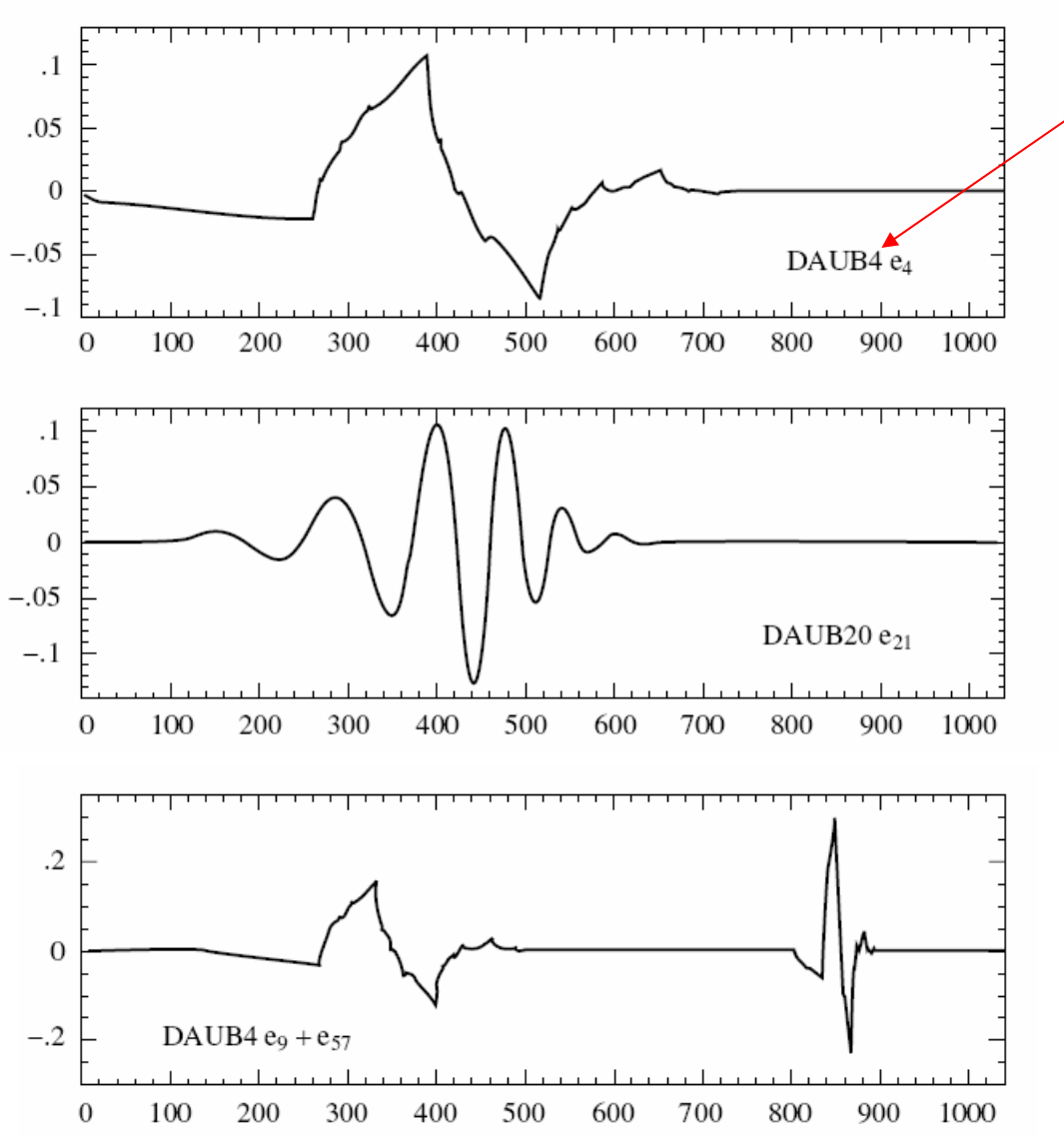
Compare to what we started with (noisy)



Of course, we can never get the original back:
information is truly lost in the presence of noise

The moral about Wiener filters is that they work in any basis, but are better in some than in others. That is what signal processing is all about!

Want to see some wavelets? Where do they come from?



The "DAUB" wavelets are named after Ingrid Daubechies, who discovered them.

(This is like getting the sine function named after you!)

So who <u>is</u> the sine function named after? it's the literal translation into Latin, ca. 1500s, of the corresponding mathematical concept in Arabic, in which language the works of Hipparchus (~150 BC) and Ptolemy (~100 AD) were preserved. The tangent function wasn't invented until the 9th Century, in Persia.